Stable and unstable del Pezzo surfaces

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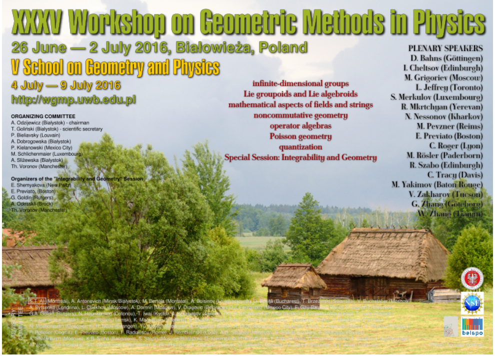


Fano manifolds

Let X be a complex projective manifold of dimension n. Put

$$-K_X = \bigwedge^n T_X.$$

Definition

The manifold X is a *Fano* manifold if $-K_X$ is an *ample* line bundle.

Example

The projective space $\mathbb{P}^n$ is a Fano manifold.

Example

The Grassmannian $\mathrm{Gr}(r, m)$ is a Fano manifold.

Example

Let X be a smooth hypersurface in $\mathbb{P}^m$ of degree d. Then

$$\text{the manifold } X \text{ is a Fano manifold} \iff d \leqslant m.$$

Low dimensional Fano manifolds

- The only 1-dimensional Fano manifold is $\mathbb{P}^1$.
- There are 10 families of 2-dimensional Fano manifolds.

Let S be a 2-dimensional Fano manifold. Put

$$\boxed{d = K_S^2.}$$

Then d is said to be the degree of the manifold S.

Theorem (Pasquale Del Pezzo)

One has

$$\boxed{d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.}$$

Moreover, one of the following cases holds:

1. $S = \mathbb{P}^2$ *and* $d = 9$*,*
2. $S = \mathbb{P}^1 \times \mathbb{P}^1$ *and* $d = 8$*,*
3. *there is a blow up* $\pi\colon S \to \mathbb{P}^2$ *at* $9 - d \leqslant 8$ *points.*

3-dimensional Fano manifolds are found by Iskovskikh, Mori, Mukai.

Del Pezzo surfaces

Definition

2-dimensional Fano manifolds are called *del Pezzo* surfaces.

Let S_d be a del Pezzo surface of degree d.

- S_9 is the projective plane $\mathbb{P}^2$.
- S_8 is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_1$.
- S_7 is the blow up of the plane $\mathbb{P}^2$ in 2 different points.
- S_6 is the blow up of the plane $\mathbb{P}^2$ in 3 different points.
- S_5 is a section of the Grassmannian

$$\mathrm{Gr}(2,5) \subset \mathbb{P}^9$$

  in its Plücker embedding by a linear space of codimension 4.
- S_4 is a complete intersection of two quadrics in $\mathbb{P}^4$.
- S_3 is a cubic surface in $\mathbb{P}^3$.
- S_2 is a quartic surface in $\mathbb{P}(1,1,1,2)$.
- S_1 is a sextic surface in $\mathbb{P}(1,1,2,3)$.

The surface S_d is toric $\iff d \geqslant 6 \iff \mathrm{Aut}(S_d)$ is infinite.

Kähler–Einstein del Pezzo surfaces

Let S_d be a del Pezzo surface of degree d.

- The Hirzebruch surface $\mathbb{F}_1$ is not Kähler–Einstein.
- The surface S_7 is not Kähler–Einstein.

Theorem (Tian, 1990)

All other del Pezzo surfaces are Kähler–Einstein.

Example

The Fubini–Study metric on $\mathbb{P}^2$ is Kähler–Einstein.

Theorem (Tian, 1987)

Let X be a Fano manifold of dimension n. If

$$\alpha(X) > \frac{n}{n+1},$$

then X is Kähler–Einstein, where $\alpha(X)$ is the α-invariant.

By Fujita, $\alpha(X) \geqslant \frac{n}{n+1}$ also implies that X is Kähler–Einstein.

Constant scalar curvature Kähler (cscK) metrics

- Let X be a complex projective manifold.
- Let L be an ample line bundle on the manifold X.

Question

When does X admit cscK metric in $c_1(L)$?

- If X is a Fano manifold and $L = -K_X$, then
  it admit cscK metric in $c_1(L) \iff X$ is Kähler–Einstein.

Conjecture (Yau, Tian, Donaldson)

X admits cscK metric in $c_1(L) \iff (X, L)$ is K-stable.

- $\Rightarrow$-direction of this conjecture is known.
- This conjecture holds for toric surfaces (Donaldson).

Theorem (Chen, Donaldson, Sun)

The conjecture holds when X is a Fano manifold and $L = -K_X$.

- Donaldson got the Breakthrough Prize for this theorem.

Ample cone of del Pezzo surfaces

Let S_d be a del Pezzo surface of degree d.

- Identify line bundles on S_d with their first Chern classes.
- Consider line bundles on S_d as classes in $H^2(S_d, \mathbb{Q}) = \mathbb{Q}^{10-d}$
- Identify $H^2(S_d, \mathbb{Q}) = H_2(S_d, \mathbb{Q})$ using Poincare pairing.

Definition

Denote by $\mathrm{Amp}(S_d)$ the cone in $H^2(S_d, \mathbb{Q})$ of ample line bundles.

The cone $\mathrm{Amp}(S_d)$ is the dual cone to the Mori cone

$$\mathrm{NE}(S_d) \subset H^2(S_d, \mathbb{Q})$$

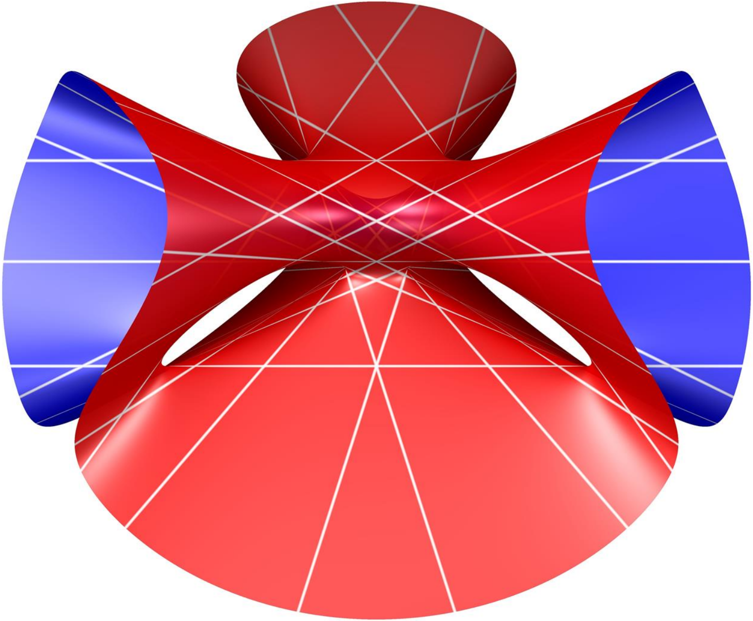
that is generated by all curves in the surface S_d.

- The cone $\mathrm{NE}(S_d)$ is a polyhedral cone.
- For $d \leqslant 7$, its rays are the curves C in S_d such that

  $$C^2 = -1$$

  and $C = \mathbb{P}^1$. Such curves are called (-1)-curves.

(-1)-curves on cubic surface



Toric del Pezzo surfaces

- Let S_d be a del Pezzo surface of degree d.
- Let L be an ample line bundle on S_d.

If S_d admits cscK metric in $\mathrm{c}_1(L)$, then $\mathrm{Aut}_0(S_d, L)$ is reductive.

Corollary

If $S_d = \mathbb{F}_1$, then S_d does not admit cscK metric in $\mathrm{c}_1(L)$.

Corollary

If $d = 7$, then S_d does not admit cscK metric in $\mathrm{c}_1(L)$.

- Recall that Fubini–Study metric is a cscK metric.

Corollary

If $S_d = \mathbb{P}^2$, then S_d admits cscK metric in $\mathrm{c}_1(L)$.

Corollary

If $S_d = \mathbb{P}^1 \times \mathbb{P}^1$, then S_d admits cscK metric in $\mathrm{c}_1(L)$.

Del Pezzo surface of degree six

- Let S_6 be a del Pezzo surface of degree 6.
- Let L be an ample line bundle on S_6.

There exists a blow up $\pi\colon S_6 \to \mathbb{P}^2$ at 3 points such that

$$\boxed{\lambda L = -K_{S_d} + a_1 E_1 + a_2 E_2 + a_3 E_3,}$$

where E_1, E_2, E_3 are π-exceptional curves, $\lambda \in \mathbb{Q}_{>0}$, and a_1, a_2, $a_3 \in \mathbb{Q}$.

Remark

The ampleness of L implies $a_i < 1$ and $a_i + a_j > -1$ for all $i \neq j$.

Theorem (Donaldson)

The surface S_6 admits cscK metric in $\mathrm{c}_1(L)$ if and only if either

$$\boxed{a_1 + a_2 + a_3 = 0}$$

or $\boxed{a_1 = a_2 = a_3}$.

Test configurations

- Let X be a complex projective manifold.
- Let L be an ample line bundle on the manifold X.

Let $\pi\colon \mathcal{X} \to \mathbb{P}^1$ be a flat morphism such that

(a) all fibers of π over $\mathbb{P}^1 \setminus [0:1]$ are isomorphic to X,

(b) the variety $\mathcal{X}$ has *mild* singularities.

Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ such that

(a) $\mathcal{L}|_X = L$, where X is any fiber of π over $\mathbb{P}^1 \setminus [0:1]$,

(b) the line bundle $\mathcal{L}$ is π-ample.

Suppose that there is an $\mathbb{C}^\star$-action on $\mathcal{X}$ such that

(a) both π and $\mathcal{L}$ are $\mathbb{C}^\star$-equivariant,

(b) the fiber of π over $[0:1]$ is $\mathbb{C}^\star$-invariant.

Then $(\mathcal{X}, \mathcal{L}, \pi\colon \mathcal{X} \to \mathbb{P}^1)$ is a *test configuration* of the pair (X, L).

Donaldson–Futaki invariant

- Let X be a complex projective manifold of dimension n.
- Let L be an ample line bundle on the manifold X. Put

$$\nu(L) = \frac{-K_X \cdot L^{n-1}}{L^n}.$$

Let $(\mathcal{X}, \mathcal{L}, \pi\colon \mathcal{X} \to \mathbb{P}^1)$ is a *test configuration* of the pair (X, L).

Definition

The *Donaldson–Futaki invariant* of $(\mathcal{X}, \mathcal{L}, \pi\colon \mathcal{X} \to \mathbb{P}^1)$ is the number

$$\boxed{\mathbf{DF} = \frac{n}{n+1}\nu(L)\mathcal{L}^{n+1} + \mathcal{L}^n \cdot \Big(K_{\mathcal{X}} - \pi^*\big(K_{\mathbb{P}^1}\big)\Big)}$$

If $\mathbf{DF} < 0$, then the pair (X, L) is said to be *K-unstable*.

Definition

The pair (X, L) is said to be *K-stable* if

- $\mathbf{DF} \geqslant 0$ for all test configurations,
- $\mathbf{DF} > 0$ for all non-isotrivial test configurations.

Unstable del Pezzo surfaces

- Let S_d be a del Pezzo surface of degree $d \geqslant 6$.
- Let L be an ample line bundle on S_d.

There exists a blow up $\pi\colon S_d \to \mathbb{P}^2$ at $r = 9 - d$ points such that

$$\boxed{\lambda L = -K_{S_d} + \sum_{i=1}^{r} a_i E_i,}$$

where $E_1, \ldots, E_r$ are π-exceptional curves, $\lambda \in \mathbb{Q}_{>0}$, and a_1, a_2, $a_3 \in \mathbb{Q}$.

Theorem (Ross, Thomas)

If $a_i \gg a_1$ for $i > 1$, then (S_d, L) is K-unstable.

- They used so-called slope-stability to prove this theorem.

Theorem (Cheltsov, Martinez-Garcia)

If $a_i \gg a_1$ and $a_i \gg a_2$ for $i > 2$, then (S_d, L) is K-unstable.

- We used slope-stability and flops to prove this theorem.

Dervan's theorem

- Let X be a Fano manifold of dimension n.
- Let L be an ample line bundle on X. Put

$$\nu(L) = \frac{-K_X \cdot L^{n-1}}{L^n} > 0.$$

Theorem (Dervan)

Suppose that $-K_X - \frac{n}{n+1}\nu(L)L$ *is numerically effective*

$$\boxed{\alpha\big(X, L\big) > \frac{n}{n+1}\nu\big(L\big).}$$

Then the pair (X, L) *is* K*-stable.*

- The α-invariant $\alpha(X, L)$ can be defined as

$$\boxed{\sup\left\{\lambda \in \mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} L\end{array}\right\}}.$$

Stable del Pezzo surfaces

- Let S_d be a del Pezzo surface of degree d.
- Let L be an ample line bundle on S_d.

Theorem (Cheltsov, Martinez-Garcia)

Suppose that $d \leqslant 2$ and

$$-K_{S_d} - \frac{2}{3}\Big(\frac{-K_{S_d} \cdot L}{L^2}\Big)L$$

is numerically effective. Then (S_d, L) is K-stable.

Example

Suppose that $d = 1$. Let E be a (-1)-curve on S_1. Put

$$L = -K_S + \lambda E,$$

where $\lambda \in \mathbb{Q}$ such that $-\frac{1}{3} < \lambda < 1$. Then (S_1, L) is K-stable if

$$3 - \sqrt{10} < \lambda < \frac{1}{9}(\sqrt{10} - 1).$$

Cubic surfaces

- Let S_3 be a smooth cubic surface in $\mathbb{P}^3$.
- Let E_1, E_2, E_3, E_4, E_5, E_6 be disjoint lines on S_3
- Let λ be a rational number. Put

$$\boxed{L = -K_S + \lambda \sum_{i=1}^{6} E_i.}$$

- Then L is ample $\iff -\frac{1}{5} \leqslant \lambda < 1$.

Theorem (Arezzo, Pacard)

Suppose that $1 > \lambda \gg 0$. *Then* (S_3, L) *is* K*-stable.*

Theorem (Cheltsov, Martinez-Garcia)

Suppose that $0 \leqslant \lambda \leqslant \frac{1}{10}$. *Then* (S_3, L) *is* K*-stable.*

- We expect that (S_3, L) is K-stable for every $\lambda \in (-\frac{1}{5}, 1)$.