The Shmushkevich method for higher symmetry groups

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Introduction

Most of the strongly interacting particles occur in sets having nearly the same mass and differing in charge by a unit steps. The origin idea was to consider the particles of a set to be the members of an isospin multiplet, where the charge states are analogous to the magnetic quantum number of a spin angular momentum multiplet.
Isospin was introduced by Heisenberg (1932) to explain a symmetry of proton and neutron.
Particle collision and decay processes probabilities were started to be calculated via Clebsch–Gordan coefficients: probability amplitudes .

60 years ago

About 60 years ago, I. Shmushkevich presented a simple ingenious method for computing relative probabilities of channels involving the same interacting multiplets of particles, without the need of computing the Clebsch-Gordan coefficients. The basic idea of Shmushkevich is the "isotopic non-polarization" or "charge-uniformity rule" of the states before the interaction and after it. His underlying Lie group was $SU(2)$.

Ex: From phenomenology ρ-mesons decay into pions (both isospin triplets)

$$\rho^+ \to \pi^+\pi^0 \ ; \ \rho^0 \to \pi^+\pi^-, \to \pi^0\pi^0 \ ; \ \rho^- \to \pi^0\pi^-$$

150 ρ-mesons (50 of each) produce 300 pions (100 of each), so the process $\rho^0 \to \pi^0\pi^0$ is forbidden.

Our approach

We extend this idea to any simple Lie group. Determination of relative probabilities of various channels of scattering and decay processes following from the invariance of the interactions with respect to a compact simple Lie group. Aiming at the probabilities and simultaneous consideration of all possible channels for given multiplets involved in the process, make the task possible. Probabilities of the states with multiplicity greater than 1 are averaged over. Examples with symmetry groups $SU(2)$,$SU(3)$, $O(5)$, $F(4)$, and $E(8)$ are shown.

Difficulties

The difficulty of the generalization of Shmushkevich's method to higher rank groups lies in the frequent occurrence of multiple states with the same quantum numbers, equivalently labeled by the same weights of an irreducible representations, as well as the sheer number of channels needed to be written down.
It is likely that practical exploitation of Shmushkevich's idea for higher groups and possibly representations of much higher dimensions, will not proceed by spelling out the large number of channels for each case and counting the number of occurrences of every state in all the channels.

Using of Weyl group and orbits

One can start from one known channel and using the Weyl group in this case, to produce other channels with the same probability. And one has to take into consideration all channels from different orbits. If the equal probability of all the states of the Lie group should be involved, the link between different orbits present in the same representation has to be imposed independently. For the probabilities a natural link is provided by the requirement that the probabilities add up to one. If an orbit is present in the irreducible representation more then once, say m times, we count them as equally probable m channels.

Simplest process

The process we consider is the simplest, where three multiplets are interacting, more precisely interaction of two particles yields a third one. Our aim is to show how to average over different particles/states, which carry the same Lie group representation labels.
In the examples we show that there are many states which have the group labels (weights) identical although their label different particle states. In order to avoid the almost impossible task to distinguish these states, we add them up and count their total probability.

Illustration : Δ baryon resonances decay

The Δ baryon resonances decay into nucleon and pion

$$\Delta^{++} \rightarrow p\pi^{+}$$

$$\Delta^{+} \rightarrow p\pi^{0} \text{ or } \rightarrow n\pi^{+}$$

$$\Delta^{0} \rightarrow n\pi^{0} \text{ or } \rightarrow p\pi^{-}$$

$$\Delta^{-} \rightarrow n\pi^{-}$$

and the isospin states correspond as follows: The triplet of π mesons π^{+}, π^{0}, π^{-} correspond to $|\,{2 \atop 2}\rangle$, $|\,{2 \atop 0}\rangle$, $|\,{2 \atop -2}\rangle$; the doublet of nucleons p and n correspond to $|\,{1 \atop 1}\rangle$ and $|\,{1 \atop -1}\rangle$and the quadruplet of resonances $\Delta^{++}, \Delta^{+}, \Delta^{0}$ and Δ^{-} with $|\,{3 \atop 3}\rangle, |\,{3 \atop 1}\rangle, |\,{3 \atop -1}\rangle$ and $|\,{3 \atop -3}\rangle$

SU(2) representation

The decay schemes we can rewrite

$$
\begin{aligned}
\left|\begin{smallmatrix}3\\3\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3\\1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3\\-1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2\\-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3\\-3\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2\\-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle
\end{aligned}
$$

Clebsch-Gordan coefficients for quadruplet decay

Applying the L_- operator on the first relation, one can express G. - C. coefficients

$$
\begin{aligned}
\left|{3 \atop 3}\right\rangle &= \left|{2 \atop 2}\right\rangle \left|{1 \atop 1}\right\rangle \\
\left|{3 \atop 1}\right\rangle &= \sqrt{\frac{2}{3}} \left|{2 \atop 0}\right\rangle \left|{1 \atop 1}\right\rangle + \sqrt{\frac{1}{3}} \left|{2 \atop 2}\right\rangle \left|{1 \atop -1}\right\rangle \\
\left|{3 \atop -1}\right\rangle &= \sqrt{\frac{2}{3}} \left|{2 \atop 0}\right\rangle \left|{1 \atop -1}\right\rangle + \sqrt{\frac{1}{3}} \left|{2 \atop -2}\right\rangle \left|{1 \atop 1}\right\rangle \\
\left|{3 \atop -3}\right\rangle &= \left|{2 \atop -2}\right\rangle \left|{1 \atop -1}\right\rangle,
\end{aligned}
$$

and the corresponding probabilities are the squares of amplitudes.

Amplitudes and cross sections

Cross sections - probabilities of canals - squares of amplitudes
Probabilities:

- $\Delta^{++} \rightarrow p\pi^{+}$ is 1
- $\Delta^{+} \rightarrow p\pi^{0}$ is $\frac{2}{3}$ and $\Delta^{+} \rightarrow n\pi^{+}$ is $\frac{1}{3}$
- $\Delta^{0} \rightarrow n\pi^{0}$ is $\frac{2}{3}$ and $\Delta^{+} \rightarrow p\pi^{-}$ is $\frac{1}{3}$
- $\Delta^{-} \rightarrow n\pi^{-}$ is 1

The Shmushkevich approach

Letus have 75 decays of each type. " Unpolarized" process means, that on the r.h.s. must be the same number of each type of particles. The 300 events produce 300 nucleons and 300 pions - 150 p, 150 n, 100 π^+, 100 π^0 and 100 π^-. Let α, β, γ and δ be positive integers

$$75\Delta^{++} \to 75p\pi^+$$

$$75\Delta^{+} \to 50p\pi^0 \text{ and } 25n\pi^+$$

$$75\Delta^{0} \to 50n\pi^0 \text{ and } 25p\pi^-$$

$$75\Delta^{-} \to 75n\pi^-,$$

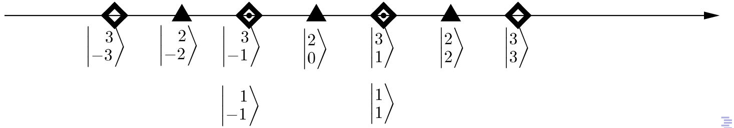
i. e. the same probabilities.

$$(1) \times (2) = (3) + (1)$$

dimensions

$$2 \times 3 = 4 + 2$$

The weight diagram of A1. Corresponding weights, Weyl group



Doublet

There are an doublet of orthogonal vectors to the quadruplet

$$\left|\begin{smallmatrix}3\\1\end{smallmatrix}\right\rangle = \sqrt{\frac{2}{3}}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle + \sqrt{\frac{1}{3}}\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}3\\-1\end{smallmatrix}\right\rangle = \sqrt{\frac{2}{3}}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle + \sqrt{\frac{1}{3}}\left|\begin{smallmatrix}2\\-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle$$

corresponds to IRR (1) with the same quantum numbers

$$\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle = \sqrt{\frac{1}{3}}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle - \sqrt{\frac{2}{3}}\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle = \sqrt{\frac{1}{3}}\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle - \sqrt{\frac{2}{3}}\left|\begin{smallmatrix}2\\-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle$$

Particle collisions channels

$$\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle = \sqrt{\frac{2}{3}}\left|\begin{smallmatrix}3\\1\end{smallmatrix}\right\rangle + \sqrt{\frac{1}{3}}\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle = \sqrt{\frac{1}{3}}\left|\begin{smallmatrix}3\\-1\end{smallmatrix}\right\rangle + \sqrt{\frac{2}{3}}\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle = \sqrt{\frac{1}{3}}\left|\begin{smallmatrix}3\\1\end{smallmatrix}\right\rangle - \sqrt{\frac{2}{3}}\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle = \sqrt{\frac{2}{3}}\left|\begin{smallmatrix}3\\-1\end{smallmatrix}\right\rangle - \sqrt{\frac{1}{3}}\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}3\\3\end{smallmatrix}\right\rangle$$

$$\left|\begin{smallmatrix}2\\-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1\\-1\end{smallmatrix}\right\rangle = \left|\begin{smallmatrix}3\\-3\end{smallmatrix}\right\rangle$$

Decomposition of the product in the A_1 example

A_1	decomposition into irreducible representations with multiplicities			multiplicity	orbit size
$(2) \times (1)$	(3)	(1)			
3×2	$[3]$			1	2
	$[1]$	$[1]$		2	2
	$[3]$	$2[1]$		6	
	decomposition into orbits with multiplicities				

Table: Decomposition of the product in the A_1 example. Decomposition is given in weight system of irreducible representations (the first line) and in terms of orbits (bottom line). The dimensions of the representations and sizes of the orbits are shown together with multiplicities.

Probabilities

Let collide all particles with the same probability. The probability of resulting particle corresponding to the weight 1 or -1 is twice then the particle corresponding to the weight 3 or -3. both pairs are in the same orbit of Weyl group, but 1 appears two times. All this orbits are of size 2.

- 2 states of [3], each present once - $\frac{1}{6}$
- 2 states of [1], each present twice - $\frac{2}{6}$

Symmetry group $SU(3)$

Consider the example where the underlying symmetry group is the Weyl reflection group of the Lie group $SU(3)$, equivalently of the Lie algebra A_2. We label the representations by their unique highest weight (relative to the basis of fundamental weights). The product of representations of dimensions 6 and 3 decomposes as follows,

$$\begin{aligned}(20) \times (10) &= (30) + (11),\\ 6 \times 3 &= 10 + 8,\end{aligned}$$

where the second line shows the dimensions of representations. Labeling the Weyl group orbits by their unique dominant weights, the product of the weight systems decomposes into the Weyl group orbits as follows,

$$\begin{aligned}(20) \times (10) &= [30] + 2[11] + 3[00],\\ 6 \times 3 &= 3 + 2 \cdot 6 + 3 \cdot 1 = 18,\end{aligned}$$

where the integers in front of square brackets are the multiplicities of occurrence of the respective orbits in the decomposition.

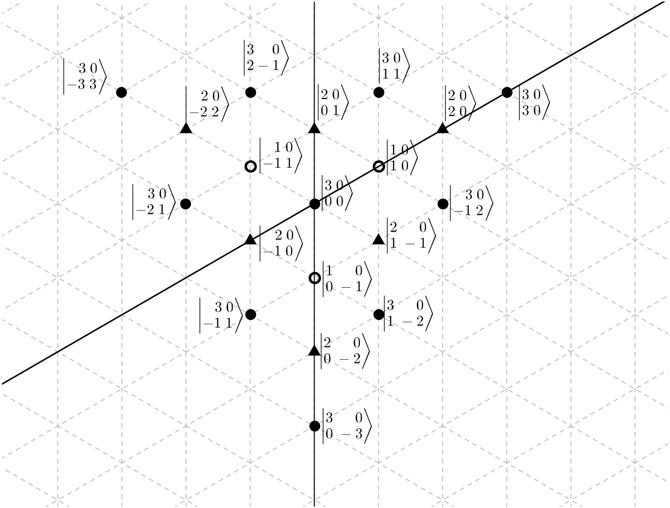

3 0
−3 3
3 0
2 − 1
3 0
1 1
2 0
−2 2
2 0
0 1
2 0
2 0
3 0
3 0
1 0
−1 1
1 0
1 0
3 0
−2 1
3 0
0 0
3 0
−1 2
2 0
−1 0
2 0
1 − 1
1 0
0 − 1
3 0
−1 1
3 0
1 − 2
2 0
0 − 2
3 0
0 − 3

Figure:

Decouplet (30)

$$
\begin{aligned}
\left|\begin{smallmatrix}3&0\\3&0\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\2&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle\\
\left|\begin{smallmatrix}3&0\\-3&3\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\-2&2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle\\
\left|\begin{smallmatrix}3&0\\0&-3\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\0&-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle\\
\left|\begin{smallmatrix}3&0\\0&0\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\1&-1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\-1&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle\\
\left|\begin{smallmatrix}3&0\\1&1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\2&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle\\
\left|\begin{smallmatrix}3&0\\2&-1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\2&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\1&-1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle\\
\left|\begin{smallmatrix}3&0\\-1&2\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\-2&2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle\\
\left|\begin{smallmatrix}3&0\\1&-2\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\0&-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\1&-1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle\\
\left|\begin{smallmatrix}3&0\\-2&1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\-2&2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\-1&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle\\
\left|\begin{smallmatrix}3&0\\-1&-1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\0&-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\-1&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle
\end{aligned}
$$

Probabilities of decay (30)

$$
\begin{aligned}
(30) \text{ - } 10 \times 30 \text{ events} &= (20) \text{ - } 6 \times 50 \text{ events}, (10) \text{ - } 3 \times 100 \text{ events} \\
\left|\begin{smallmatrix}3&0\\3&0\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\2&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3&0\\-3&3\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\-2&2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3&0\\0&-3\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\0&-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3&0\\0&0\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\1&-1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\-1&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3&0\\1&1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\2&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3&0\\2&-1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\2&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\1&-1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3&0\\-1&2\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\-2&2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3&0\\1&-2\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\0&-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\1&-1\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3&0\\-2&1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\-2&2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\-1&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle \\
\left|\begin{smallmatrix}3&0\\-1&-1\end{smallmatrix}\right\rangle &= \left|\begin{smallmatrix}2&0\\0&-2\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right\rangle \text{ or } \left|\begin{smallmatrix}2&0\\-1&0\end{smallmatrix}\right\rangle\left|\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right\rangle
\end{aligned}
$$

Decomposition of the product in the A_2 example

A_2	decomposition into irreducible representations with multiplicities			multiplicity	orbit size
$(20) \times (10)$	(30)	(11)			
6×3	[30]			1	3
	[11]	[11]		2	6
	[00]	2[00]		3	1
	[30]	2[11]	3[00]	18	
	decomposition into orbits with multiplicities				

Table: Decomposition of the product in the A_2 example. Decomposition is given in weight system of irreducible representations (the first line) and in terms of orbits (bottom line). The dimensions of the representations and sizes of the orbits are shown together with multiplicities.

Probabilities

Let collide all particles with the same probability. The probability of resulting particle corresponding to the weights (11), (2-1),(-21),(-11),(1-2) or (-12) is twice then the particle corresponding to the weight (30), (-33) or (0-3), and the probability of (00) is three times then the previous.

- 3 states of [30], each present once - $\frac{1}{18}$
- 6 states of [11], each present twice - $\frac{1}{9}$
- 1 states of [00] presents three times - $\frac{1}{6}$

Symmetry group $O(5)$

Consider the example where the underlying symmetry group is the Weyl reflection group of the Lie group $O(5)$, equivalently of the Lie algebra C_2. We label the representations by their unique highest weight (relative to the basis of fundamental weights). The product of representations of dimensions 10 and 4 decomposes as follows,

$$\begin{aligned}(20) \times (10) &= (30) + (11) + (10),\\ 10 \times 4 &= 20 + 16 + 4 = 40,\end{aligned}$$

where the second line shows the dimensions of representations. Labeling the Weyl group orbits by their unique dominant weights, the product of the weight systems decomposes into the Weyl group orbits as follows,

$$\begin{aligned}(20) \times (10) &= [30] + 2[11] + 5[10],\\ 10 \times 4 &= 4 + 2 \cdot 8 + 5 \cdot 4 = 40,\end{aligned}$$

where the integers in front of square brackets are the multiplicities of occurrence of the respective orbits in the decomposition.

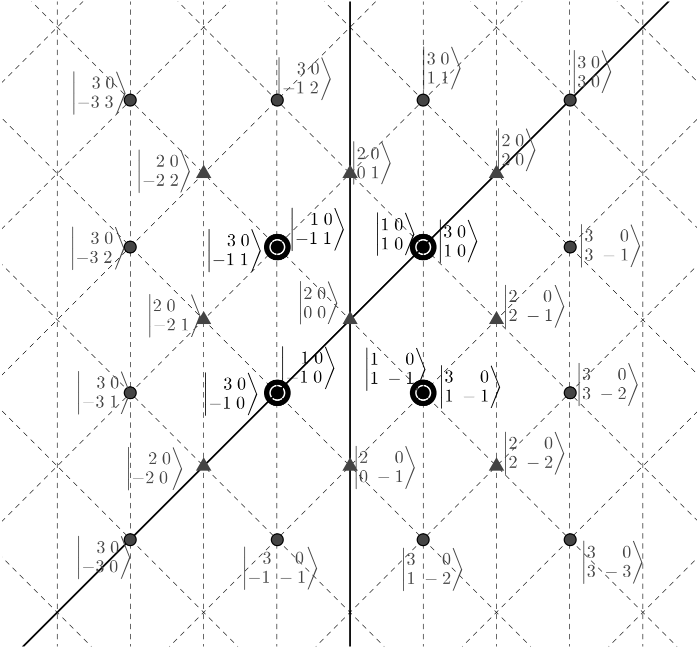

3 0 −3 3
3 0 −1 2
3 0 1 1
3 0 3 0
2 0 −2 2
2 0 0 1
2 0 2 0
3 0 −3 2
3 0 −1 1
1 0 −1 1
1 0 1 0
3 0 1 0
3 0 3 −1
2 0 −2 1
2 0 0 0
2 0 2 −1
3 0 −3 1
3 0 −1 0
1 0 −1 0
1 0 1 −1
3 0 1 −1
3 0 3 −2
2 0 −2 0
2 0 0 −1
2 0 2 −2
3 0 −3 0
3 0 −1 −1
3 0 1 −2
3 0 3 −3

Figure:

C_2	decomposition into irreducible representations with multiplicities			multiplicity	orbit size
$(20) \times (10)$	(30)	(11)	(10)		
10×4	$[30]$			1	4
	$[11]$	$[11]$		2	8
	$2[10]$	$2[10]$	$[10]$	5	4
	$[30]$	$2[11]$	$5[10]$	40	
	decomposition into orbits with multiplicities				

Table: Decomposition of the product in the C_2 example. Decomposition is given in weight system of irreducible representations (the first line) and in terms of orbits (bottom line). The dimensions of the representations and sizes of the orbits are shown together with multiplicities.

There are 40 states in the product. If equal probability is assumed, each of the channels comes with the probability $\frac{1}{40}$. Consequently, we have the probabilities

- 4 states from [30], each present once – $\frac{1}{40}$
- 8 states from [11], each present twice – $\frac{1}{20}$;
- 4 states from [10], each present 5 times – $\frac{1}{8}$.

Symmetry group $F(4)$

Consider decomposition of the product of representations in terms of their weight systems,

$$\begin{aligned}(0001) \times (0001) &= (0002) + (0010) + (1000) \\ &\quad + (0001) + (0000) \\ 26 \times 26 &= 324 + 273 + 52 + 26 + 1 = 676.\end{aligned}$$

Decomposition of the same product in terms of Weyl group orbits,

$$\begin{aligned}(0001) \times (0001) &= [0002] + 2[0010] + 6[1000] \\ &\quad + 12[0001] + 28[0000]\end{aligned}$$

and the corresponding equality of dimensions

$$26 \times 26 = 24 + 2 \cdot 96 + 6 \cdot 24 + 12 \cdot 24 + 28 \cdot 1.$$

$$(0001) \times (0001)$$

(0002)	(0010)	(1000)	(0001)	(0000)	size
[0002]					24
[0010]	[0010]				96
3[1000]	2[1000]	[1000]			24
5[0001]	5[0001]	[0001]	[0001]		24
12[0000]	9[0000]	4[0000]	2[0000]	[0000]	1
[0002]	2[0010]	6[1000]	12[0001]	28[0000]	$676 = 26^2$

The last row shows the multiplicity of each orbit in decomposition.

If the equal probability of the 676 states is assumed with have the following probabilities of the channels

- 24 states from [0002], each present once $\frac{1}{676}$;
- 96 states from [0010], each present twice $\frac{2}{676}$;
- 24 states from [1000], each present 6 times $\frac{6}{676}$;
- 24 states from [0001], each present 12 times $\frac{12}{676}$;
- 1 states from [0000], each present 28 times $\frac{28}{676}$.

Symmetry group $E(8)$

Consider decomposition of the product of representations in therms of their weight systems

$$\left(\begin{smallmatrix}&&&&0&&\\1&0&0&0&0&0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}&&&&0&&\\1&0&0&0&0&0&0\end{smallmatrix}\right)=\left(\begin{smallmatrix}&&&&0&&\\2&0&0&0&0&0&0\end{smallmatrix}\right)+\left(\begin{smallmatrix}&&&&0&&\\0&1&0&0&0&0&0\end{smallmatrix}\right)+\left(\begin{smallmatrix}&&&&0&&\\0&0&0&0&0&0&1\end{smallmatrix}\right)+\left(\begin{smallmatrix}&&&&0&&\\1&0&0&0&0&0&0\end{smallmatrix}\right)+\left(\begin{smallmatrix}&&&&0&&\\0&0&0&0&0&0&0\end{smallmatrix}\right)$$

with the respective dimensions

$$248\times 248=27000+30380+3875+248+1=61504.$$

The same product decomposed into the sum of Weyl group orbits has very different multiplicities,

$$\left(\begin{smallmatrix}&&&&0&&\\1&0&0&0&0&0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}&&&&0&&\\1&0&0&0&0&0&0\end{smallmatrix}\right)=\left[\begin{smallmatrix}&&&&0&&\\2&0&0&0&0&0&0\end{smallmatrix}\right]+2\left[\begin{smallmatrix}&&&&0&&\\0&1&0&0&0&0&0\end{smallmatrix}\right]+14\left[\begin{smallmatrix}&&&&0&&\\0&0&0&0&0&0&1\end{smallmatrix}\right]+72\left[\begin{smallmatrix}&&&&0&&\\1&0&0&0&0&0&0\end{smallmatrix}\right]+304\left[\begin{smallmatrix}&&&&0&&\\0&0&0&0&0&0&0\end{smallmatrix}\right],$$

and the equality of the dimensions in the decomposed product,

$$248\times 248=240+2\cdot 6720+14\cdot 2160+72\cdot 240+304\cdot 1=61504.$$

$$\binom{0}{1000000} \times \binom{0}{1000000}$$

$\binom{0}{2000000}$	$\binom{0}{0100000}$	$\binom{0}{0000001}$	$\binom{0}{1000000}$	$\binom{0}{0000000}$	size
$\left[{0 \atop 2000000}\right]$					240
$\left[{0 \atop 0100000}\right]$	$\left[{0 \atop 0100000}\right]$				6720
$6\left[{0 \atop 0000001}\right]$	$7\left[{0 \atop 0000001}\right]$	$\left[{0 \atop 0000001}\right]$			2160
$29\left[{0 \atop 1000000}\right]$	$35\left[{0 \atop 1000000}\right]$	$7\left[{0 \atop 1000000}\right]$	$\left[{0 \atop 1000000}\right]$		240
$120\left[{0 \atop 0000000}\right]$	$140\left[{0 \atop 0000000}\right]$	$35\left[{0 \atop 0000000}\right]$	$8\left[{0 \atop 1000000}\right]$	$\left[{0 \atop 0000000}\right]$	1
$\left[{0 \atop 2000000}\right]$	$2\left[{0 \atop 0100000}\right]$	$14\left[{0 \atop 0000001}\right]$	$72\left[{0 \atop 1000000}\right]$	$304\left[{0 \atop 0000000}\right]$	248^2

The last row shows the multiplicity of each orbit in decomposition.

If equal probability of the 61504 states is assumed we have the following probabilities of the channels

- 240 states from $\left[\begin{smallmatrix}0\\2000000\end{smallmatrix}\right]$, each present once $\frac{1}{61504}$;
- 6720 states from $\left[\begin{smallmatrix}0\\0100000\end{smallmatrix}\right]$, each present twice $\frac{2}{61504}$;
- 2160 states from $\left[\begin{smallmatrix}0\\0000001\end{smallmatrix}\right]$, each present 14 times $\frac{14}{61504}$;
- 240 states from $\left[\begin{smallmatrix}0\\1000000\end{smallmatrix}\right]$, each present 72 times $\frac{72}{61504}$;
- 1 states from $\left[\begin{smallmatrix}0\\0000000\end{smallmatrix}\right]$, it presents 304 times $\frac{304}{61504}$;

THANK YOU FOR THE ATTENTION