Noncommutative gauge theory of generalized (quantum) Weyl algebras

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WGMP XXXV, 2016

References:

TB, *Noncommutative differential geometry of generalized Weyl algebras*, SIGMA 12 (2016) 059.
TB, *Circle and line bundles over generalized Weyl algebras*, Algebr. Represent. Theory 19 (2016), 57–69.

Aims:

- To construct (modules of sections of) cotangent and spinor bundles over noncommutative surfaces (generalized Weyl algebras).
- To construct real spectral triples (Dirac operators) on noncommutative surfaces.

The classical construction

- Let M be a surface.
- Construct a principal bundle

$$\begin{tikzcd} P \arrow[d, "\pi"] & U(1) \arrow[l] \\ M & \end{tikzcd}$$

  such that T^*P is a trivial bundle, and
- $$T^*M \cong P \times_{U(1)} V,$$

  as (non-trivial) vector bundles, and
- $$SM \cong P \times_{U(1)} W,$$

  as (trivial) vector bundles.
- Example: $M = S^2$, $P = S^3$.

Algebraically

We need to consider:

- an algebra $\mathcal{B}$ (of smooth functions on M),
- an algebra $\mathcal{A}$ (of smooth functions on P).
- P is an $U(1)$-principal bundle over M means that $\mathcal{A}$ is strongly graded by $\mathbb{Z}$, the Pontrjagin dual of $U(1)$, and $\mathcal{B}$ is isomorphic to the degree-zero part of $\mathcal{A}$.

Further we need:

- A first-order differential calculus $\Omega\mathcal{A}$ on $\mathcal{A}$ (sections of T^*P) such that $\Omega\mathcal{A}$ is free as a left and right $\mathcal{A}$-module (triviality of T^*P).
- Restriction of $\Omega\mathcal{A}$ to a calculus $\Omega\mathcal{B}$ on $\mathcal{B}$.
- Identification of $\Omega\mathcal{B}$ in terms of sums of homogeneous parts of $\mathcal{A}$ (sections of $T^*M \cong P \times_{U(1)} V$) .
- A candidate for a Dirac operator from the canonical connection on $\mathcal{A}$.

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Principal bundles vs. strongly graded algebras

- Let G be a compact Lie group and M a compact manifold.
- A compact manifold P is a principal G-bundle over M provided that G acts freely on P and $M \cong P/G$.
- If G is abelian, freeness of action on M is equivalent to the strong grading of the algebra of functions on P by the Pontrjagin dual of G.
- $U(1)$-principal bundles correspond to strongly $\mathbb{Z}$-graded (commutative) algebras.
- Noncommutative $U(1)$-principal bundles $\equiv$ strongly $\mathbb{Z}$-graded (noncommutative) algebras.

Strongly graded algebras

- Let G be a group. An algebra $\mathcal{A}$ is *G-graded* if

$$\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g, \qquad \mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh}, \quad \forall g, h \in G.$$

- $\mathcal{A}$ is *strongly G-graded* provided, for all $g, h \in G$,

$$\mathcal{A}_g \mathcal{A}_h = \mathcal{A}_{gh}$$

- Strong grading is equivalent to the existence of a mapping

$$\ell : G \to \mathcal{A} \otimes \mathcal{A},$$

  such that

$$\ell(g) \in \mathcal{A}_{g^{-1}} \otimes \mathcal{A}_g, \quad m(\ell(g)) = 1.$$

- ℓ is called a *strong connection*.

Strongness of the $\mathbb{Z}$-grading

- A $\mathbb{Z}$-graded algebra $\mathcal{A}$ is strongly graded if and only if there exist

$$\omega = \sum_i \omega'_i \otimes \omega''_i \in \mathcal{A}_{-1} \otimes \mathcal{A}_1, \quad \bar{\omega} = \sum_i \bar{\omega}'_i \otimes \bar{\omega}''_i \in \mathcal{A}_1 \otimes \mathcal{A}_{-1},$$

such that

$$\sum_i \omega'_i \omega''_i = \sum_i \bar{\omega}'_i \bar{\omega}''_i = 1.$$

- Construct inductively elements: $\ell(n) \in \mathcal{A}_{-n} \otimes \mathcal{A}_n$ as

$$\ell(0) = 1 \otimes 1, \qquad \ell(n) = \begin{cases} \sum_i \omega'_i \ell(n-1) \omega''_i & \text{if } n > 0, \\ \sum_i \bar{\omega}'_i \ell(n+1) \bar{\omega}''_i & \text{if } n < 0. \end{cases}$$

Strong $\mathbb{Z}$-connections and idempotents

- In a strongly $\mathbb{Z}$-graded algebra $\mathcal{A}$, $\mathcal{A}_n$ are projective (invertible) modules over $\mathcal{B} = \mathcal{A}_0$; they are modules of sections of line bundles associated to $\mathcal{A}$.
- Write $\ell(n) = \sum_{i=1}^{N} \ell'(n)_i \otimes \ell''(n)_i$.
- Form an $N \times N$-matrix $E(n)$ with entries

$$E(n)_{ij} = \ell''(n)_i \ell'(n)_j.$$

- $E(n)$ is an idempotent for $\mathcal{A}_n$.

Algebras we want to study: Quantum surfaces

- Let p be a polynomial in one variable such that $p(0) \neq 0$ and $q \in \mathbb{K}$, $k \in \mathbb{N}$.
- $\mathcal{B}(p; q, k)$ denotes the algebra generated by x, y, z subject to relations:

$$xz = q^2 zx, \qquad yz = q^{-2} zy,$$

$$xy = q^{2k} z^k p(q^2 z), \qquad yx = z^k p(z).$$

- The algebras $\mathcal{B}(p; q, k)$ have GK-dimension 2, and hence can be understood as coordinate algebras of noncommutative surfaces.
- If $\mathbb{K} = \mathbb{C}$ and p has real coefficients, then $\mathcal{B}(p; q, k)$ is a $*$-algebra by $y = x^*$, $z = z^*$.

Examples of quantum surfaces

- The Podleś sphere: $k = 1$, $p(z) = 1 - z$.
- The noncommutative torus: $k = 0$, $p(z) = 1$.
- The quantum disc: $k = 0$, $p(z) = 1 - z$.
- Set:

$$p(z) = \prod_{l=0}^{N-1} \left(1 - q^{-2l} z\right).$$

Then

(a) $k = 0$ – quantum cones,

(b) $k = 1$ – quantum teardrops,

(c) $k > 1$ – quantum spindles.

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  - (c) $k > 1$ – quantum spindles.

Algebras we want to study: Total spaces

- Let p be a polynomial, $p(0) \neq 0$ and $q \in \mathbb{K}$, $k \in \mathbb{N}$.
- Let $\mathcal{A}(p;q)$ be generated by $x_\pm, z_\pm$ subject to relations:

$$z_+z_- = z_-z_+, \quad x_+z_\pm = q^{-1}z_\pm x_+, \quad x_-z_\pm = qz_\pm x_-,$$

$$x_+x_- = p(z_+z_-), \qquad x_-x_+ = p(q^2z_-z_+).$$

- View it as a $\mathbb{Z}$-graded algebra with degrees of $z_\pm$ being equal to ± 1, and that of $x_\pm$ being equal to $\pm k$.
- Define

$$\mathcal{A}(p;q,k) := \bigoplus_{n\in\mathbb{Z}} \mathcal{A}(p;q)_{nk},$$

- Note that $\mathcal{A}(p;q,1) = \mathcal{A}(p;q)$ with $x_\pm$ given degrees ± 1.
- If $\mathbb{K} = \mathbb{C}$ and p is real then $\mathcal{A}(p;q,k)$ is a $*$-algebra via $z_\pm^* = z_\mp$, $x_\pm^* = x_\mp$.

Examples of $\mathcal{A}(p; q)$

- $\mathcal{O}(SU_q(2))$: $p(z) = 1 - z$.
- Quantum lens spaces :

$$p(z) = \prod_{l=0}^{N-1} \left(1 - q^{-2l} z\right).$$

Generalized Weyl algebras

- [Bavula] Let $\mathcal{R}$ be an algebra, σ an automorphism of $\mathcal{R}$ and p an element of the centre of $\mathcal{R}$. A *degree-one generalized Weyl algebra over* $\mathcal{R}$ is an algebraic extension $\mathcal{R}(p,\sigma)$ of $\mathcal{R}$ obtained by supplementing $\mathcal{R}$ with additional generators X, Y subject to the following relations

$$XY = \sigma(p), \quad YX = p, \quad Xa = \sigma(a)X, \quad Ya = \sigma^{-1}(a)Y.$$

- The algebras $\mathcal{R}(p,\sigma)$ share many properties with $\mathcal{R}$, in particular, if $\mathcal{R}$ is a Noetherian algebra, so is $\mathcal{R}(p,\sigma)$, and if $\mathcal{R}$ is a domain and $p \neq 0$, so is $\mathcal{R}(p,\sigma)$.
- $\mathcal{A}(p;q)$, $\mathcal{B}(p;q,k)$ are examples of generalized Weyl algebras (over $\mathcal{R}[z_+,z_-]$ and $\mathcal{R}[z]$, respectively).

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Quantum principal bundles over quantum surfaces

Theorem

View $\mathcal{A}(p; q, k)$ as a $\mathbb{Z}$-graded algebra by considering $a \in \mathcal{A}(p; q, k)$ to be of degree n if it has a degree kn in $\mathcal{A}(p; q)$. Then

(1) $\mathcal{B}(p; q, k) \cong \mathcal{A}(p; q, k)_0$, by identification $x := x_- z_+^k$, $y := z_-^k x_+$ and $z := z_+ z_-$.

(2) $\mathcal{A}(p; q, k)$ is a strongly $\mathbb{Z}$-graded algebra.

Differential calculi

- A *first-order differential calculus* on $\mathcal{A}$ is an $\mathcal{A}$-bimodule $\Omega\mathcal{A}$ with a $\mathbb{K}$-linear map $d : \mathcal{A} \to \Omega\mathcal{A}$ such that
  (a) d satisfies the Leibniz rule: for all $a, b \in \mathcal{A}$,
  $$d(ab) = d(a)b + ad(b);$$
  (b) $\Omega\mathcal{A}$ satisfies the *density condition*: $\Omega\mathcal{A} = \mathcal{A}d(\mathcal{A})$.
- If $\mathcal{B} \subset \mathcal{A}$ is a subalgebra, then one can restrict $\Omega\mathcal{A}$ to
  $$\Omega\mathcal{B} := \mathcal{B}d(\mathcal{B})\mathcal{B}.$$
- If $\mathcal{A}$ is a complex $*$-algebra, then the calculus $(\Omega\mathcal{A}, d)$ is said to be a $*$-*calculus* provided $\Omega\mathcal{A}$ is equipped with an anti-linear operation $*$ such that, for all $a, b \in \mathcal{A}$, $\omega \in \Omega\mathcal{A}$,
  $$(a\omega b)^* = b^*\omega^*a^* \quad \text{and} \quad d(a^*) = d(a)^*.$$

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Skew derivations

- Noncommutative vector fields do not normally satisfy the Leibniz rule, but often they do satisfy the *skew* Leibniz rule.
- By a *skew σ-derivation on $\mathcal{A}$* we mean a pair (∂, σ), where σ is an algebra automorphism of $\mathcal{A}$ and $\partial : \mathcal{A} \to \mathcal{A}$ is a linear map such that, for all $a, b \in \mathcal{A}$,

$$\partial(ab) = \partial(a)\sigma(b) + a\partial(b);$$

Differential calculi from skew derivations

- Fix a finite indexing set I, and let (∂_i, σ_i), $i \in I$, be a collection of skew derivations on an algebra $\mathcal{A}$.
- Let $\Omega\mathcal{A}$ be a free left $\mathcal{A}$-module with a free basis ω_i, $i \in I$.
- Define the (free) right $\mathcal{A}$-module structure on $\Omega\mathcal{A}$ by setting

$$\omega_i a := \sigma_i(a)\omega_i.$$

- Then the map

$$d : \mathcal{A} \to \Omega\mathcal{A}, \qquad a \mapsto \sum_{i \in I} \partial_i(a)\omega_i,$$

satisfies the Leibniz rule.
- There is no guarantee in general that the density condition be satisfied.

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Skew derivations on $\mathcal{A}(p;q,1)$

Theorem

Let, for all $a \in \mathcal{A}(p;q,1)$,

$$\sigma_\pm(a) = q^{|a|}a, \quad \sigma_0(a) = q^{2|a|}a, \quad c(z) := q\frac{p(q^2z) - p(z)}{(q^2-1)z}.$$

For all $\alpha_{0,\pm} \in \mathbb{K}$, the maps $\partial_{0,\pm}$ defined on the generators of $\mathcal{A}(p;q,1)$ by

$$\partial_0(x_+) = \alpha_0 x_+, \quad \partial_0(x_-) = -q^{-2}\alpha_0 x_-,$$

$$\partial_0(z_+) = \alpha_0 z_+, \quad \partial_0(z_-) = -q^{-2}\alpha_0 z_-,$$

and

$$\partial_\mp(x_\pm) = \partial_\mp(z_\pm) = 0, \quad \partial_\mp(x_\mp) = \alpha_\mp c(z) z_\pm, \quad \partial_\mp(z_\mp) = \alpha_\mp x_\pm;$$

extend to the whole of $\mathcal{A}(p;q,1)$ as skew $\sigma_{0,\pm}$-derivations.

Differential calculus on $\mathcal{A}(p; q, 1)$

Theorem

If $q^2 \neq 1$ and $p(z) \neq 0$ is coprime with $p(q^2 z)$, then the system of skew-derivations (∂_i, σ_i), $i \in \{+, -, 0\}$, defines the first-order differential calculus $\Omega\mathcal{A}$ on $\mathcal{A}(p; q, 1)$ with free generators ω_+, ω_-, ω_0 and differential

$$d(a) = \partial_-(a)\omega_- + \partial_0(a)\omega_0 + \partial_+(a)\omega_+.$$

In the case of $p(z) = 1 - z$, with properly chosen constants α_i, $\Omega\mathcal{A}$ is the (left-covariant) 3D calculus on the quantum group $SU_q(2)$ introduced by Woronowicz.

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Differential calculus on $\mathcal{B}(p; q, 1)$

Theorem

(1) *For all* $a \in \mathcal{B}(p; q, 1)$*,*

$$\partial_0(a) = 0.$$

(2) *If* $q^4 \neq 1$ *and* $p(z) \neq 0$ *is coprime with* $p(q^2 z)$*, then*

$$\Omega\mathcal{B} \cong \mathcal{A}(p; q, 1)_{-2} \oplus \mathcal{A}(p; q, 1)_2,$$

where $\Omega\mathcal{B}$ *is the restriction of* $\Omega\mathcal{A}$ *to the calculus on* $\mathcal{B}(p; q, 1)$*.*

(3) *The cotangent bundle over* $\mathcal{B}(p; q, 1)$ *is non-trivial, as the module of sections* $\Omega\mathcal{B}$ *is not free.*

The real spectral triple for $\mathcal{B}(p; q, 1)$

- A Dirac operator on $\mathcal{B}(p; q, 1)$ is constructed by following the procedure of Beggs and Majid '15.
- The sections of a spinor bundle are identified with the $\mathcal{B}(p; q, 1)$-bimodule $\mathcal{A}(p; q, 1)_1 \oplus \mathcal{A}(p; q, 1)_{-1}$,

$$\mathcal{S}_+ = \mathcal{A}(p; q, 1)_{-1}\mathrm{s}_+, \quad \mathcal{S}_- = \mathcal{A}(p; q, 1)_1\mathrm{s}_-, \quad \mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-,$$

- As there are idempotents $E(1)$ and $E(-1)$ such that $E(1) + E(-1) = 1$, the spinor bundle is trivial.
- Note that, individually, $\mathcal{S}_-$ and $\mathcal{S}_+$ are not trivial.

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The real spectral triple for $\mathcal{B}(p;q,1)$

- The strong connection forms $\ell(1)$, $\ell(-1)$ define a connection $\nabla : \mathcal{S} \to \Omega\mathcal{B} \otimes \mathcal{S}$ on the spinor bundle $\mathcal{S}$ by the formula

$$\nabla(a\,\mathrm{s}_+ + b\,\mathrm{s}_-) = \pi(d(a))\ell(-1)\,\mathrm{s}_+ + \pi(d(b))\ell(1)\,\mathrm{s}_-,$$

  for all $a, b \in \mathcal{A}(p;q,1)$, a of degree -1 and b of degree 1. Here π is the projection of $\Omega\mathcal{A}$ onto horizontal forms

$$\mathcal{A}(p;q,1)d(\mathcal{B}(p;q,1))\mathcal{A}(p;q,1) = \mathcal{A}(p;q,1)\omega_+ \oplus \mathcal{A}(p;q,1)_-\omega_-.$$

- The Clifford action $\triangleright$ of $\Omega\mathcal{B}$ on $\mathcal{S}$ is defined, for all $a, b, c_\pm \in \mathcal{A}(p;q,1)$ of degrees $|a| = -1$, $|b| = 1$, $|c_\pm| = \pm 2$, by

$$(c_-\omega_+ + c_+\omega_-)\triangleright(a\,\mathrm{s}_+ + b\,\mathrm{s}_-) = \beta_+ c_- b\,\mathrm{s}_+ + \beta_- c_+ a\,\mathrm{s}_-,$$

  where $\beta_+, \beta_- \in \mathbb{K}$

The real spectral triple for $\mathcal{B}(p;q,1)$

- The Dirac operator given by

$$D := \triangleright \circ \nabla : \mathcal{S} \to \mathcal{S},$$

  comes out as

$$D(a\,s_+ + b\,s_-) = \beta_+ q^{-1} \partial_+(b)\, s_+ + \beta_- q \partial_-(a) s_-.$$

- D is an even Dirac operator with the grading

$$\gamma : \mathcal{S} \to \mathcal{S}, \qquad a\,s_+ + b\,s_- \longmapsto a\,s_+ - b\,s_-.$$

The real spectral triple for $\mathcal{B}(p; q, 1)$

Theorem

Let $\mathbb{K} = \mathbb{C}$, $q \in (0, 1)$ and p be a q^2-separable polynomial with real coefficients. Choose $\beta_\pm$ such that $\beta_-^*/\beta_+ < 0$, and let ν be a solution to the equation

$$\nu^2 = -q^3 \frac{\beta_-^*}{\beta_+}.$$

Then the linear map

$$J : \mathcal{S} \to \mathcal{S}, \qquad a\,s_+ + b\,s_- \longmapsto -\nu^{-1} b^* s_+ + \nu a^* s_-,$$

equips D with a real structure such that D has KO-dimension two.