Deformed Fock spaces, Hecke operators and monotone Fock space of Muraki

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Plan of the presentation

1. Fock spaces of Yang-Baxter type
   - Hecke operator
   - Positivity of T-symmetrizators
2. Hecke operators and positivity of T-symmetrizators
   - Positivity of $P_T^{(n)}$ for Yang-Baxter-Hecke operators
3. Connections of Woronowicz-Pusz operators T_{μ}^{CAR}
4. Non-commutative Levy process for generalized "ANYON"

In this talk we will present the following subjects:

1. Fock spaces of Yang-Baxter type.
    - Hecke operators and
    - positivity of T-symmetrizators.
2. Connections of Woronowicz-Pusz operators T_{μ}^{CAR} with monotone Fock space of Muraki-Lu ($\mu = 0$).
3. Non-commutative Levy process for generalized "ANYON" statistics.

Fock spaces of Yang-Baxter type

Let $T : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ be a Yang-Baxter operator, i.e.

$$T_1 T_2 T_1 = T_2 T_1 T_2, \quad T = T^*, \quad T \geq -\mathrm{I}$$

on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, where $T_1 = T \otimes \mathrm{I}$, $T_2 = \mathrm{I} \otimes \mathrm{T}$. We define the T-symmetrizator operator

$$P_T^{(n)}(T_1, T_2, \ldots, T_{n-1}) = P_T^{(n)} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$$

as follows:

$$P_T^{(n)} = (1 + T_1 + T_2 T_1 + T_3 T_2 T_1 + \ldots$$

$$\ldots + T_{n-1} \ldots T_1) P_T^{(n-1)}(T_2, T_3, \ldots, T_{n-1}),$$

where $P_T^{(1)} = 1$, $P_T^{(2)} = 1 + T_1$ and

$$T_i = \underbrace{1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes T \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i-1 \text{ times}} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}.$$

Under the assumption that $\|T\| \leq 1$ we proved (see Bożejko, Speicher 1991, 1994) ,
that $P_T^{(n)} \geq 0$ for each n and hence we can form a new pre-scalar product on $\mathcal{H}^{\otimes n}$ as follows: for $\xi, \eta \in \mathcal{H}^{\otimes n}$

$$\langle \xi | \eta \rangle_T := \langle P_T^{(n)} \xi | \eta \rangle,$$

where $\langle \cdot | \cdot \rangle$ is the natural scalar product on $\mathcal{H}^{\otimes n}$.

Then we can form the creation operator

$$a^{+}(f)\xi = f \otimes \xi$$

and the annihilation operator

$$a(f)\xi = l(f)(1 + T_1 + T_1 T_2 + \ldots + T_1 T_2 \ldots T_{n-1})\xi \qquad \text{for } \xi \in \mathcal{H}^{\otimes n},$$

where $l(f)$ is the free annihilation operator defined as

$$l(f)(x_1 \otimes \ldots \otimes x_n) = \langle f | x_1 \rangle \; x_2 \otimes \ldots \otimes x_n.$$

The main object of this note is the structure of the von Neumann algebra

$$\Gamma_T(\mathcal{H}) = \{G_T(f) : f \in \mathcal{H}_{\mathbb{R}}\}''$$

generated by the T-Gaussian field $G_T(f) = a^{+}(f) + a(f)$, where $\mathcal{H}_{\mathbb{R}}$ denotes the real part of $\mathcal{H}$.

As was shown in many papers- Voiculescu, Bożejko-Speicher, Ricard, Nou
that this von Neumann algebra is a non-injective II_1-factor.
The linear span of T-Gaussian random variables is completely isomorphic to the operator space called row and column, as we will show later.
This is an extension of the results of Haagerup and Pisier , A. Buchholz and of our results with R.Speicher.

Now we solve the question posed by L. Accardi, when the T-symmetrizator operators $P_T^{(n)}$ are similar to self-adjoint projections, i.e.

$$\left(P_T^{(n)}\right)^2 = \alpha(n)\ P_T^{(n)} \qquad \text{for some } \alpha(n) > 0. \tag{1}$$

First, let us see that if $P_T^{(2)} = 1 + T$ satisfies (1) then

$$(1+T)^2 = \alpha(1+T) \qquad \text{for } \alpha = \alpha(2) \tag{2}$$

which implies that

$$T^2 = (q-1)T + q\ 1, \tag{3}$$

where $q = \alpha - 1$. Such an operator satisfying (3) is called *Hecke operator* with parameter q.

Examples of Hecke operators

:

(H_1) The flip $T = \sigma : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ given by an exchange of the factors $\sigma(x \otimes y) = y \otimes x$ is a Hecke operator with $q = 1$ and the corresponding "projection"

$$P_T^{(n)} = \sum_{\pi \in S_n} \pi$$

is the classical symmetrizator operator on $\mathcal{H}^{\otimes n}$.

(H_2) For $T = -\sigma$ we obtain the anti-symmetrizator

$$P_T^{(n)} = \sum_{\pi \in S_n} \mathrm{sgn}(\pi)\ \pi,$$

where $\mathrm{sqn}(\pi)$ is the classical sign of a permutation $\pi \in S_n$.

(H_3) If we take $\epsilon = \pm 1$ and we define the operator

$$T = T_\epsilon = \frac{q-1}{2} + \epsilon \frac{q+1}{2} \sigma$$

then we get the *Hecke operator* with parameter q, i.e.

$$T^2 = (q-1)T + q\ 1.$$

This operator is a Yang-Baxter operator if and only if $q = 1$, which means that T_ϵ is the symmetrizator ($\epsilon = 1$) or the anti-symmetrizator ($\epsilon = -1$).

(H_4) We get a very interesting example of a Yang-Baxter-Hecke operator for a Hilbert space $\mathcal{H}$ of finite dimension $\dim \mathcal{H} = m$ with an orthonormal basis $(e_1, e_2, \ldots, e_m)$. We consider the operator $\tilde{P} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ given by

$$\tilde{P}(e_i \otimes e_j) = -\frac{1}{m}\delta_{ij} \sum_{k=1}^{m} e_k \otimes e_k .$$

One can see that $P = (-\tilde{P})$ is the projector operator of the following form:

$$P(x \otimes y) = \frac{1}{m}\langle x|y\rangle \ \theta, \qquad \text{where } \theta = \sum_{k=1}^{m} e_k \otimes e_k, \quad x, y \in \mathcal{H}$$

(see Goodman,Wallach book, page 449).

(H_5) The main example of that talk is Pusz-Woronowicz twisted CCR (CAR) operators: T_μ^{CCR}, T_μ^{CAR} defined as :

$$T_\mu^{CCR}(e_i \otimes e_j) = \begin{cases} \mu(e_j \otimes e_i) & \text{if } i < j, \\ \mu^2(e_i \otimes e_i) & \text{if } i = j, \\ -(1-\mu^2)(e_i \otimes e_j) + \mu(e_j \otimes e_i) & \text{if } i > j, \end{cases}$$

$$T_\mu^{CAR}(e_i \otimes e_j) = \begin{cases} -\mu(e_j \otimes e_i) & \text{if } i < j, \\ -(e_i \otimes e_i) & \text{if } i = j, \\ -(1-\mu^2)(e_i \otimes e_j) - \mu(e_j \otimes e_i) & \text{if } i > j. \end{cases}$$

Both the twisted CCR and twisted CAR are Yang-Baxter-Hecke operators with the parameter $q = \mu^2$, which means that

$$T^2 = (\mu^2 - 1)T + \mu^2\ 1.$$

(H_6) As a special case we consider $T_0^{CAR} = T^M$, where T^M is of the following form:

$$T^M(e_i \otimes e_j) = \begin{cases} 0 & \text{if } i < j, \\ -(e_i \otimes e_j) & \text{if } i \geq j. \end{cases}$$

It is connected with Muraki-Lu monotone Fock space, as we will see later.

(H_7) Also it will be interesting to see the corresponding T-Fock space in the case when the twisted CCR operator has parameter $\mu = 0$ and then we get the following operator:

$$T_0^{CCR}(e_i \otimes e_j) = \begin{cases} 0 & \text{if } i \leq j, \\ -(e_i \otimes e_j) & \text{if } i > j. \end{cases}$$

Later we will use this operator to construct the Bose monotone Fock space.

(H_8) In the paper Bożejko, Lytvynov and Wysoczanski , Comm.Math.Phys.2012, we introduced another type (called anyonic) of the Yang-Baxter-Hecke operator T_z on $L^2(\mathbb{R},\sigma)$, where σ is a non-atomic Radon measure on $\mathbb{R}$ defined for $f \in L^2(\mathbb{R}^2,\sigma\otimes\sigma)$ as follows:

$$T_z f(x,y) = Q(x,y) f(y,x),$$

where $|z| = 1$ and

$$Q(x,y) = \begin{cases} z & \text{if } x < y, \\ \bar{z} & \text{if } x > y. \end{cases}$$

Then T_z is a Yang-Baxter-Hecke operator with parameter $q = 1$.

Positivity of $P_T^{(n)}$ for Yang-Baxter-Hecke operators

Proposition 1.

Let $T = T^*$ be a Yang-Baxter-Hecke operator. Then for each $n \geq 1$

$$\left(P_T^{(n)}\right)^2 = \underline{n}!\ P_T^{(n)} \geq 0, \qquad (*)$$

where $\underline{n} = 1 + q + \ldots + q^{n-1}$ and $\underline{n}! = \underline{1} \cdot \underline{2} \cdot \ldots \cdot \underline{n}$.
Moreover, for $q \geq -1$

$$\left\| P_T^{(n)} \right\| = \underline{n}! = \frac{\prod_{k=1}^{n}(1-q^k)}{(1-q)^n}.$$

Remark

Proposition 1 solves the problem of L. Accardi: $P_T^{(n)}$ is similar to a projection if and only if T is a Hecke operator.

T-symmetric Fock Hilbert space

$$\mathcal{F}_T(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circledast n} \underline{n}! = \mathbb{C}\Omega \oplus \mathcal{H} \oplus \mathcal{H}^{\circledast 2} \oplus \cdots ,$$

where

$$\mathcal{H}^{\circledast n} = P_T^{(n)}(\mathcal{H}^{\otimes n})$$

is the space of T-symmetric tensors. By Proposition 1 we have that for $f \in \mathcal{H}^{\circledast n}$, $P_T^{(n)}(f) = \underline{n}! f$. Therefore the Hilbert norm $\|f\|_T^2$ for $f = (f_0, f_1, f_2, \dots) \in \mathcal{F}_T(\mathcal{H})$ is defined as:

$$\|f\|_T^2 = \langle P_T(f)|f\rangle = \sum_{n=0}^{\infty} \langle P_T^{(n)}(f_n)|f_n\rangle = \sum_{n=0}^{\infty} \underline{n}! \ \|f_n\|^2 \leq \infty.$$

One can see that we have the following description of T-symmetric tensors:

Lemma

For $n > 1$ we have

$$\mathcal{H}^{\circledast n} = \left\{ f \in \mathcal{H}^{\otimes n} : T_j(f) = qf \text{ for } j \in \{1, 2, \dots, n-1\} \right\} = \left\{ f \in \mathcal{H}^{\otimes n} : \tilde{P}_T^{(n)}(f) = f \right\},$$

where $\tilde{P}_T^{(n)} = \frac{1}{\underline{n}!} P_T^{(n)}$.

Remark

Let us observe that the T-creation and T-annihilation operators on the T-Fock space can be defined as follows: for $f \in \mathcal{H}$

$$a_T^+(f) = \tilde{P}\ l^+(f)\tilde{P} = \tilde{P}\ l^+(f),$$

$$a_T(f) = \tilde{P}\ l(f)\tilde{P} = l(f)\ \tilde{P},$$

where $\tilde{P} = \tilde{P}_T = \sum_{n=0}^{\infty} \frac{1}{n!} P_T^{(n)}$ is the orthogonal projection onto T-symmetric tensors and $l^+(f), l(f)$ are the free creation and free annihilation operators.

Boolean Fock spaces

The simplest among deformed T-symmetric Fock spaces is the Boolean Fock space $\mathcal{F}_{-1}(\mathcal{H}) = \mathbb{C}\Omega \oplus \mathcal{H}$ and the Yang-Baxter-Hecke operator $T = -I$, $P_T^{(n)} = 0$ for $n > 1$.
The Boolean creation and annihilation operators are following:

$$b^+(f)\xi = \begin{cases} 0 & \text{if } \xi \in \mathcal{H}, \\ f & \text{if } \xi = \Omega, \end{cases}$$

$$b(f)\xi = \begin{cases} \langle f|\xi\rangle & \text{if } \xi \in \mathcal{H}, \\ 0 & \text{if } \xi \in \mathbb{C}\Omega. \end{cases}$$

They satisfy the following relations: if $(e_1, e_2, \ldots, e_N)$ is an orthonormal basis of $\mathcal{H}$ and $b_i^\pm := b^\pm(e_i)$ then

$$b_i b_j^+ = \delta_{i,j}\left(1 - \sum_{k=0}^{N} b_k^+ b_k\right) = \delta_{i,j} P_\Omega,$$

For the Boolean Gaussian random variables

$$G^B(f) = b(f) + b^+(f),$$

the following Proposition is known to be true:

Proposition 2.(Bożejko, Krystek, Wojakowski , 2006)

For arbitrary operators $\alpha_i \in B(\mathcal{H})$ and $f_i \in \mathcal{H}_{\mathbb{R}}$, $\|f_i\| = 1$, we have

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes G^B(f_i) \right\| = \max \left\{ \left\| \left(\sum_{i=1}^{N} \alpha_i \alpha_i^* \right)^{1/2} \right\|, \left\| \left(\sum_{i=1}^{N} \alpha_i^* \alpha_i \right)^{1/2} \right\| \right\}.$$

That means that Boolean Gaussian random variables span the operator space completely isometrically isomorphic to row and column operator space. Similar results was obtained for the free and q-Gaussian random variables and free generators (see Haagerup-Pisier, Bożejko-Speicher).

Monotone Fock spaces

Now we recall the definition of the monotone Fock space following the fundamental paper of Muraki ,1996, and we show that it is equal to special case of the T-symmetric Fock space for the Pusz-Woronowicz operator considered in the example (H_6)
$T_0^{CAR} = T^M$.
Let $\mathbb{N}$ be the set of all natural numbers. For $r \geq 1$ we define
$I_r = \{(i_1, i_2, \ldots, i_r) : i_1 < i_2 < \cdots < i_r, \; i_j \in \mathbb{N}\}$ and for $r = 0$ we set $I_0 = \{\emptyset\}$, where $\emptyset$ denotes the null sequence.
We define $\mathrm{Inc}(\mathbb{N}) = \bigcup_r I_r$. Let $\mathcal{H}_r = l^2(I_r)$ be the r-particle Hilbert space and $\Phi = \bigoplus_{r=0}^{\infty} \mathcal{H}_r$ the monotone Fock space.

For an increasing sequence $\sigma = (i_1, i_2, \dots, i_r) \in \mathrm{Inc}(\mathbb{N})$, denote by $[\sigma] = \{i_1, i_2, \dots, i_r\}$ the associated set and by $\{e_\sigma\}$ the cannonical basis vector in the monotone Fock space Φ.
We will write $[\sigma] < [\tau]$ if for each $i \in [\sigma]$ and $j \in [\tau]$ we have $i < j$.
The monotone creation operator δ_i^+ and the annihilation operator δ_i^- are defined for each $i \in \mathbb{N}$ by:

$$\delta_i^+ e_{(i_1,\dots,i_r)} = \begin{cases} e_{(i,i_1,\dots,i_r)} & \text{if } \{i\} < \{i_1, \dots, i_r\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_i^- e_{(i_1,\dots,i_r)} = \begin{cases} e_{(i_2,\dots,i_r)} & \text{if } r \geq 1, i = i_1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us observe that if $P = \bigoplus P^{(n)}$ is the orthogonal projection from the full Fock space onto the monotone Fock space, then $\delta_i^{\pm} = P l_i^{\pm} P$, where $l_i^{\pm}$ are the free creation and the free annihilation operators. Moreover, the following relations hold:

$$\begin{aligned}
\delta_i^{+}\delta_j^{+} = \delta_j^{-}\delta_i^{-} &= 0 && \text{for } i \geq j,\\
\delta_i^{-}\delta_j^{+} &= 0 && \text{for } i \neq j,\\
\delta_i^{-}\delta_i^{+} &= 1 - \sum_{j \leq i} \delta_j^{+}\delta_j^{-} && \text{for } i = j.
\end{aligned}$$

Proposition 3.

If $T^M = T_0^{CAR}$ is the Pusz-Woronowicz Yang-Baxter-Hecke operator defined as

$$T^M(e_i \otimes e_j) = \begin{cases} 0 & \text{if } i < j, \\ -(e_i \otimes e_j) & \text{if } i \geq j \end{cases}$$

then the T-symmetric Fock space is exactly Muraki monotone Fock space and the corresponding creation and annihilation operators are the following:

$$a_i^+ = \delta_i^+, \qquad a_i = \delta_i^-.$$

Proposition 4.

Let $\alpha_i \in B(\mathcal{H})$ and $G_i = \delta_i^- + \delta_i^+$ be the monotone Gaussian operators. Then

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^- \right\| = \left\| \sum_{i=1}^{N} \alpha_i \alpha_i^* \right\|^{1/2}, \tag{4}$$

$$\left\| \sum_{i=1}^{N} \alpha_i \otimes \delta_i^+ \right\| = \left\| \sum_{i=1}^{N} \alpha_i^* \alpha_i \right\|^{1/2}, \tag{5}$$

$$\|\delta_i^-\| = \|\delta_i^+\| = 1, \tag{6}$$

$$1 \leq \|G_i\| \leq 2, \tag{7}$$

$$\max\left\{\left\|\left(\sum_{i=1}^{N}\alpha_i\alpha_i^*\right)^{1/2}\right\|,\left\|\left(\sum_{i=1}^{N}\alpha_i^*\alpha_i\right)^{1/2}\right\|\right\}\leq$$
$$\left\|\sum_{i=1}^{N}\alpha_i\otimes G_i\right\|\leq 2\max\left\{\left\|\left(\sum_{i=1}^{N}\alpha_i\alpha_i^*\right)^{1/2}\right\|,\left\|\left(\sum_{i=1}^{N}\alpha_i^*\alpha_i\right)^{1/2}\right\|\right\}. \tag{8}$$

Bose Monotone Fock spaces

If we consider bosonic type of the operator Pusz-Woronowicz defined as

$$T^B(e_i \otimes e_j) = T_0^{CCR}(e_i \otimes e_j) = \begin{cases} 0 & \text{if } i \leq j, \\ -(e_i \otimes e_j) & \text{if } i > j \end{cases}$$

then one can see that in that case the n-th particle space of the corresponding T-symmetric Fock space $\mathcal{F}_{T^B}(\mathcal{H})$ is of the following form:

$$P_T^{(n)}(\mathcal{H}^{\otimes n}) = \mathcal{H}_T^{\otimes n} = \mathrm{Lin}\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} : i_1 \leq i_2 \leq \cdots \leq i_n\}.$$

The action of the creation and annihilation operators is following:

$$\Delta_j^+(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = \begin{cases} e_j \otimes e_{i_j} \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} & \text{if } j \leq i_1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Delta_j(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = \begin{cases} e_{i_2} \otimes \cdots \otimes e_{i_n} & \text{if } j = i_1, \\ 0 & \text{otherwise.} \end{cases}$$

and they satisfy the following commutation relations:

$$\begin{aligned} \Delta_i \Delta_j^+ =& 0 && \text{for } i \neq j, \\ \Delta_i \Delta_i^+ =& 1 - \sum_{k<i} \Delta_k^+ \Delta_k && \text{for } i = j. \end{aligned}$$

From the last formulas we get $\Delta_1 \Delta_1^+ = 1$ and $\|\Delta_1\| = 1$.
Moreover, $\|\Delta_j^\pm\| \leq 1$ and since $\|\Delta_j^+ \Omega\| = 1$ we have $\|\Delta_j^\pm\| = 1$.

By the similar considerations like in the Fermi-monotone Muraki-Fock space we have the following :

Proposition 5.

Let $\alpha_i \in B(\mathcal{H})$ and $g_i = \Delta_i^- + \Delta_i^+$ be the monotone Bose Gaussian operators, then

$$\max\left\{\left\|\left(\sum_{i=1}^{N}\alpha_i\alpha_i^*\right)^{1/2}\right\|,\left\|\left(\sum_{i=1}^{N}\alpha_i^*\alpha_i\right)^{1/2}\right\|\right\} \leq$$

$$\left\|\sum_{i=1}^{N}\alpha_i\otimes g_i\right\| \leq 2\max\left\{\left\|\left(\sum_{i=1}^{N}\alpha_i\alpha_i^*\right)^{1/2}\right\|,\left\|\left(\sum_{i=1}^{N}\alpha_i^*\alpha_i\right)^{1/2}\right\|\right\}.$$

If we take the vacuum state $\varepsilon(T) = \langle T\Omega, \Omega\rangle$, then one can show the following central limit theorem for the Bose-monotone Gaussian random variables $g_i = \Delta_i + \Delta_i^+$.

Proposition 6.(Central Limit Theorem, 2000.)

If $S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} g_i$, then

$$\lim_{N\to\infty} \varepsilon(S_N^{2n}) = \binom{2n}{n},$$

i.e., S_N weakly tends to arcsine law $\frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}$.

In the case of the Fermi monotone case this same law was obtained by N. Muraki (1996). See also the paper of J. Wysoczanski(2011) for related generalization of the central limit theorems for the Boolean-monotonic case.

When we will consider the Pusz-Woronowicz Hecke operator T_μ^{CCR} for $\mu = -1$, defined as

$$T^B(e_i \otimes e_j) = T_{-1}^{CCR}(e_i \otimes e_j) = \begin{cases} (e_i \otimes e_j) & \text{if } i = j, \\ -(e_i \otimes e_j) & \text{if } i \neq j. \end{cases}$$

we get the model of mixed Bose-Fermi commutation relations:

$$b_i b_j^+ + b_j^+ b_i = 0 \text{ if } \ i \neq j,$$
$$b_i b_j^+ - b_j^+ b_i = 1 \text{ if } \ i = j.$$

That models correspond to so called q_{ij}-CCR commutations relations of the form

$$A_i A_j^+ - q_{ij} A_j^+ A_i = \delta_i j 1,$$

where $q_{ij} = \bar{q}_{ji}$ and $|q_{ij}| \leq 1$.

Such models were considered in many papers: Bożejko, Speicher, Jorgensen, Smith, Werner, Nou, Krolak, Yoshida, Hiai, Lust-Piquard, Śniady.
In our last case we have case of "anicommuting bosons" i.e.:
$q_{ii} = 1$ and $q_{ij} = -1$ for $i \neq j$.
Similarly, if we consider the Pusz-Woronowicz-Hecke operator T_{μ}^{CAR}, for $\mu = -1$,
we obtain again q_{ij}-CCR commutation relations of the type "commuting fermions", when $q_{ii} = -1$ and $q_{ij} = 1$ for $i \neq j$.

Non-commutative Levy process for generalized "ANYON"

A first rigorous interpolation between canonical commutation relations (CCR) and canonical anticommutation relations (CAR) was constructed in 1991 by Bożejko and Speicher . Given a Hilbert space $\mathcal{H}$, we constructed, for each $q \in (-1, 1)$, a deformation of the full Fock space over $\mathcal{H}$, denoted by $\mathcal{F}^q(\mathcal{H})$. For each $h \in \mathcal{H}$, one naturally defines a (bounded) creation operator, $a^+(h)$, in $\mathcal{F}^q(\mathcal{H})$. The corresponding annihilation operator, $a^-(h)$, is the adjoint of $a^+(h)$. These operators satisfy the q-commutation relations:

$$a^-(g)a^+(h) - qa^+(h)a^-(g) = (g, h)_{\mathcal{H}}, \quad g, h \in \mathcal{H}.$$

This is special case of Yang-Baxter deformation given by the $T_q(x \otimes y) = q(y \otimes x)$.
The limiting cases, $q = 1$ and $q = -1$, correspond to the Bose and Fermi statistics, respectively.

Another generalization of the CCR and CAR was proposed in 1995 by Ligouri and Mintchev . They fixed a *continuous* underlying space $X = \mathbb{R}$ and considered a function $Q : X^2 \to \mathbb{C}$ satisfying $Q(s,t) = \overline{Q(t,s)}$ and $|Q(s,t)| = 1$.
Setting $\mathcal{H}$ to be the complex space $L^2(R)$, one defines a bounded linear operator acting on $\mathcal{H} \otimes \mathcal{H}$ by the formula

$$T(f \otimes g)(s,t) = Q(s,t)g(s)f(t), \quad f,g \in \mathcal{H}. \tag{9}$$

This operator is self-adjoint, its norm is equal to 1, and it satisfies the Yang-Baxter and Hecke relation.

One then defines corresponding creation and annihilation operators, $a^{+}(h)$ and $a^{-}(h)$, for $h \in \mathcal{H}$. By setting $a^{+}(h) = \int_T dt\, h(t)\partial_t^{\dagger}$ and $a^{-}(h) = \int_T dt\, \overline{h(t)}\partial_t$,
one gets (at least informally) creation and annihilation operators, $\partial_t^{\dagger}$ and ∂_t, at point $t \in T$.
These operators satisfy the Q-commutation relations

$$\partial_s\partial_t^{\dagger} - Q(s,t)\partial_t^{\dagger}\partial_s = \delta(s,t),$$
$$\partial_s\partial_t - Q(t,s)\partial_t\partial_s = 0, \quad \partial_s^{\dagger}\partial_t^{\dagger} - Q(t,s)\partial_t^{\dagger}\partial_s^{\dagger} = 0. \tag{10}$$

From the point of view of physics, the most important case of a generalized statistics (10) is the anyon statistics. For the anyon statistics, the function Q is given by

$$Q(s,t) = \begin{cases} q, & \text{if } s < t, \\ \bar{q}, & \text{if } s > t \end{cases}$$

for a fixed $q \in \mathbb{C}$ with $|q| = 1$. Hence, the commutation relations (10) become

$$\partial_s \partial_t^\dagger - q \partial_t^\dagger \partial_s = \delta(s,t),$$
$$\partial_s \partial_t - \bar{q} \partial_t \partial_s = 0, \quad \partial_s^\dagger \partial_t^\dagger - \bar{q} \partial_t^\dagger \partial_s^\dagger = 0, \qquad (11)$$

for $s < t$.

The free Levy processes, i.e. case $Q(s,t) = q = 0$, was done in our paper with E.Lytvynov.

Having creation, neutral, and annihilation operators at our disposal, we define and study, a noncommutative stochastic process (white noise)

$$\omega(t) = \partial_t^{\dagger} + \partial_t + \lambda \partial_t^{\dagger} \partial_t, \quad t \in T.$$

Here $\lambda \in \mathbb{R}$ is a fixed parameter. The case $\lambda = 0$ corresponds to a Q-analog of Brownian motion, while the case $\lambda \neq 0$ (in particular, $\lambda = 1$) corresponds to a (centered) Q-Poisson process .
We identify corresponding Q-Hermite (Q-Charlier respectively) polynomials, denoted by $\omega(t_1) \cdots \omega(t_n)$, of infinitely many noncommutative variables $(\omega(t))_{t \in T}$.

Then we introduced the notion of independence for a generalized statistics, and to derive corresponding Lévy processes. We know from experience both in free probability and in q-deformed probability that a natural way to explain that certain noncomutative random variables are independent (relative to a given statistics/deformation of commutation relations) is to do this through corresponding deformed cumulants-like q-deformed cumulants $(-1 < q < 1)$.

Noncommutative Lévy processes have most actively been studied in the framework of free probability. Using q-deformed cumulants, Anshelevich constructed and studied noncommutative Lévy processes for q-commutation relations.

Main references:

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