The Realization Formula(e)

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Thanks!

Thanks to the organizers for this opportunity and warm hospitality.

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We become more ambitious.

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Let there be a Hilbert space H and a contraction

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H$$

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The surprise is that all H^∞ functions on the unit disc with sup norm not more than 1 is of this form: a function f is in in $H^\infty(\mathbb{D})$ and satisfies $\|f\| \leq 1$ if and only if there is a Hilbert space H and a contraction (iff an isometry iff a unitary operator)

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such that $f(z) = A + zB(I - zD)^{-1}C$.

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This technique is known as that of a lurking isometry.

It was shown by Agler that a function f is in $H^\infty(\mathbb{D}^2)$ and has norm less than or equal to 1 if and only if there is a graded Hilbert space $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ and a unitary operator $V : \mathbb{C} \oplus \mathcal{L} \to \mathbb{C} \oplus \mathcal{L}$ which can be written in the block form as

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

so that writing P_1 for projection of $\mathcal{L}$ onto $\mathcal{L}_1$ and P_2 for projection of $\mathcal{L}$ onto $\mathcal{L}_2$,

$$f((z) = A + B(z_1 P_2 + z_2 P_2)[I - (z_1 P_1 + z_2 P_2)D]^{-1} C.$$

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In fact, $\beta = (s - \overline{s}p)/(1 - |p|^2)$.

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Note that the easier part of the proof of the formula for the unit disk is to show that an f of the form

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indeed has modulus no bigger than one (holomorphicity is clear).
This assertion of the modulus being not bigger than one depends on T being a contraction.
That would remain the same if the operators A, B, C and D are not constants, but operator valued functions defined on the open unit disk which satisfy the property that the 2×2 operator matrix valued function

$$T(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} : \mathbb{C} \oplus H \to \mathbb{C} \oplus H$$

is a contraction valued function.

In that case, $f(z)$ becomes

$$f(z) = A(z) + zB(z)(I - zD(z))^{-1}C(z). \tag{1}$$

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Consider, then, two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and an isometry V on $\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{K}$. Obviously, V has a block operator matrix decomposition:

$$V = \begin{pmatrix} P & Q & R \\ S & T & U \\ W & X & Y \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ \mathcal{H} \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ \mathcal{H} \\ \mathcal{K} \end{pmatrix}. \tag{2}$$

The following 2×2 principal submatrices

$$\begin{pmatrix} P & R \\ W & Y \end{pmatrix}, \begin{pmatrix} Q & R \\ X & Y \end{pmatrix}, \begin{pmatrix} S & U \\ W & Y \end{pmatrix} \text{ and } \begin{pmatrix} T & U \\ X & Y \end{pmatrix}$$

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of V are contractions.
Associate with each of them a function on $\mathbb{G}$ that depends on the submatrix and a chosen complex number α in the closed unit disk. The functions are

$$A(s,p) = P + (2\alpha p - s)R\big((2 - \alpha s) - (2\alpha p - s)Y\big)^{-1}W \quad (3)$$

$$B(s,p) = Q + (2\alpha p - s)R\big((2 - \alpha s) - (2\alpha p - s)Y\big)^{-1}X$$

$$C(s,p) = S + (2\alpha p - s)U\big((2 - \alpha s) - (2\alpha p - s)Y\big)^{-1}W$$

$$D(s,p) = T + (2\alpha p - s)U\big((2 - \alpha s) - (2\alpha p - s)Y\big)^{-1}X$$

Note that the functions have been so prepared that

$$A(s,p) \in \mathcal{B}(\mathbb{C},\mathbb{C}) \text{ and } B(s,p) \in \mathcal{B}(\mathcal{H},\mathbb{C})$$

$$C(s,p) \in \mathcal{B}(\mathbb{C},\mathcal{H}) \text{ and } D(s,p) \in \mathcal{B}(\mathcal{H},\mathcal{H}).$$

Thus if we now define the colligation associated with V and α as the 2×2 operator matrix function defined on $\mathbb{G}$ by

$$(s,p) \to T_{V,\alpha}(s,p) = \begin{pmatrix} A(s,p) & B(s,p) \\ C(s,p) & D(s,p) \end{pmatrix} \tag{4}$$

then it is a $\mathcal{B}(\mathbb{C} \oplus \mathcal{H})$ valued function on $\mathbb{G}$.

We are now ready to state the Realization Theorem. It will involve the following 2×2 diagonal operator matrix which we shall call the **evaluation operator**. For $(s,p) \in \mathbb{G}$ and two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as above, the evaluation operator is the following strict contraction on $\mathcal{H}_1 \oplus \mathcal{H}_2$:

$$\mathcal{E}(s,p) = \begin{pmatrix} \frac{s}{2} I_{\mathcal{H}_1} & 0 \\ 0 & p I_{\mathcal{H}_2} \end{pmatrix}.$$

Realization theorem. *The following are equivalent.*

H *f is a function in $H^\infty(\mathbb{G})$ with $\|f\|_\infty \le 1$.*

R *There is a complex number α in the closed unit disk, two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and an isometry*

$$V = \begin{pmatrix} P & Q & R \\ S & T & U \\ W & X & Y \end{pmatrix} : \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{K} \to \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{K} \tag{5}$$

such that the associated colligation $T_{V,\alpha}$, as defined in (4), is contraction valued and

$$f(s,p) = A(s,p) + B(s,p)\mathcal{E}(s,p)(I - D(s,p)\mathcal{E}(s,p))^{-1}C(s,p). \tag{6}$$

Γ as a spectral set

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That is so because of Oka-Weil Theorem (Γ is polynomially convex!)

Admissible kernels:

Definition

A kernel k is a scalar valued function on $\Omega \times \Omega$ which is holomorphic in the first variable, anti-holomorphic in the second variable and is positive definite, i.e., $\sum_{i,j=1}^{n} c_i \bar{c}_j k(z, z_j) > 0$ for any positive integer n, any n points $z_1, z_2, \ldots, z_n$ in Ω and any n scalars $c_1, c_2, \ldots, c_n$.

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Given such a kernel k, there is a Hilbert space of holomorphic functions H_k such that the the family of functions $\{k(\cdot, w) : w \in \Omega\}$ is contained in H_k, is dense in H_k and has the reproducing property, i.e.,

$$f(z) = \langle f, k(\cdot, z) \rangle$$

for an f in H_k and any z in Ω. Because of this reproducing property, the Hilbert space H_k is called a reproducing kernel Hilbert space.

Admissible kernels

A multiplier on the reproducing kernel Hilbert space H_k is a holomorphic function φ defined on $\mathbb{G}$ such that the multiplication operator

$$M_\varphi : f \to \varphi f$$

is a bounded operator on H_k. Of particular importance to us will be the following multipliers.

$$(M_s f)(s,p) = sf(s,p) \text{ and } (M_p f)(s,p) = pf(s,p). \qquad (7)$$

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Definition

A kernel $k(s,p)$ on $\mathbb{G}$ is called admissible if the pair of multiplication operators (M_s, M_p) on the reproducing kernel Hilbert space H_k is a Γ-contraction on H_k.

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Interpolation Theorem. *Given* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *in* $\mathbb{G}$ *and* $w_1, w_2, \ldots, w_n$ *in* $\overline{\mathbb{D}}$*, there is a function* f *in* $H^\infty(\mathbb{G})$ *with* $\|f\|_\infty \leq 1$ *and satisfying* $f(\lambda_i) = w_i, i = 1, 2, \ldots, n$ *if and only if for every admissible kernel* k*, the matrix*

$$((\ (1 - w_i \overline{w}_j) k(\lambda_i, \lambda_j) \ )) \tag{8}$$

is positive definite.

Extension

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The effort is to find a property of V that is necessary and sufficient to ensure that every bounded holomorphic function in a neighbourhood of V extends to the whole of the summetrized bidisk in such a way that the H^∞-norm of the original function on V is not increased.

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The symbol $\mathrm{Hol}^\infty(V)$ stands for those bounded functions f on V which have a holomorphic extension to a neighbourhood of V.

Let $\mathcal{A}$ be a subset of $\mathrm{Hol}^\infty(V)$. We shall explain two properties of the set V below - the $\mathcal{A}$-extension property and the property of being an $\mathcal{A}$-von Neumann set.

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The $\mathcal{A}$-extension property means that whenever $f \in \mathcal{A}$, there is a bounded holomorphic function g on whole of $\mathbb{G}$ such that

$$g|_V = f \text{ and } \sup_{\mathbb{G}} |g| = \sup_V |f|. \tag{9}$$

An extension of the form (9) is what we want to achieve, motivated by a theorem of Cartan. The challenge is to decide what kind of sets V will allow us that.

Extension......

The motivation for defining an $\mathcal{A}$-von Neumann set comes from the 1951 paper of von Neumann where he showed that for a contraction T on a Hilbert space and a polynomial p, the following inequality is satisfied.

$$\|p(T)\| \leq \sup_{z \in \mathbb{D}} |p(z)|.$$

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A dozen years later, Ando came up with an elegant generalization of this inequality. If (T_1, T_2) is a commuting pair of contractions, and p is a polynomial in two variables, then

$$\|p(T_1, T_2)\| \leq \sup_{z_1, z_2 \in \mathbb{D}} |p(z_1, z_2)|.$$

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A polynomially convex compact set $X \subseteq \mathbb{C}^2$ is called a spectral set for a pair of commuting bounded operators if $\sigma(T_1, T_2) \subseteq X$ and

$$\|p(T_1, T_2)\| \leq \sup_{X} |p|$$

If $V \subseteq \mathbb{C}^2$, say that a pair of commuting operators (T_1, T_2) on a Hilbert space is $subordinate$ to V if the Taylor joint spectrum $\sigma(T_1, T_2) \subseteq V$ and $g(T_1, T_2) = 0$ whenever g is holomorphic in a neighbourhood of V and $g|_V = 0$.

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If f is a function on V that has a holomorphic extension to a neighboruhood of V and (T_1, T_2) is subordinate to V, define $f(T_1, T_2)$ by setting

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Given $\mathcal{A}$ as above, V is called an $\mathcal{A}$-von Neumann set if for any Γ-contraction subordinate to V and any $f \in \mathcal{A}$,

$$\|f(S, P)\| \leq \sup_V |f|.$$

Let $\lambda_1 = (s_1, p_1), \lambda_2 = (s_2, p_2), \ldots, \lambda_n = (s_n, p_n)$ be n distinct points in the symmetrized bidisk $\mathbb{G}$. Let $w_1, w_2, \ldots, w_n$ be n points in $\bar{\mathbb{D}}$. A normal family argument shows that the following infimum is attained.

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$$\begin{aligned} \rho \;=\; & \inf\{\|f\|_\infty : f \text{ is a holomorphic function from } \mathbb{G} \text{ into } \bar{\mathbb{D}} \\ & \quad \text{satisfying } f(s_i, p_i) = w_i \text{ for } i = 1, 2, \ldots, n\}. \end{aligned} \tag{10}$$

A function f is called extremal if the infimum above is attained for f. A compactness argument that uses the Interpolation Theorem proves the following lemma.

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A function f is called extremal if the infimum above is attained for f. A compactness argument that uses the Interpolation Theorem proves the following lemma.

Lemma

If f is an extremal for ρ, then there is a Γ-contraction (S, P) subordinate to $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $\|f(S, P)\| = \rho$.

Extension Theorem. *Let* $V \subseteq \mathbb{G}$. *Let* $\mathcal{A} \subseteq Hol^{\infty}(V)$. *Then* V *has the* $\mathcal{A}$*-extension property if and only if* V *is an* $\mathcal{A}$*-von Neumann set.*

Thank you for your attention.