DUALITY FOR DOUBLE STRUCTURES

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Double vector bundles are implicit in the connection theory of vector bundles. A connection in a vector bundle $E \to M$ gives a lifting of vector fields on M to vector fields on E; the latter are *linear* in the sense that they are morphisms of vector bundles from E to TE with the vector bundle structure on base TM obtained by applying the tangent functor to all the vector bundle operations; this is the *tangent prolongation of* E. Connections can also be formulated as linear maps $E \times_M TM \to TE$ which combines right-inverses to both natural maps $TE \to E$ and $TE \to TM$ shown in Figure 1(a) below.

$$\begin{tikzcd} TE \arrow[r] \arrow[d] & TM \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

(a)

$$\begin{tikzcd} T^*E \arrow[r] \arrow[d] & E^* \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

(b)

$$\begin{tikzcd} T(E^*) \arrow[r] \arrow[d] & TM \arrow[d] \\ E^* \arrow[r] & M \end{tikzcd}$$

(c)

Figure 1.

Figure 1(a) shows the two vector bundle structures on TE; the standard structure with base E and the tangent prolongation with base TM. Each of these can be dualized in the usual way and they lead to the double vector bundles in (b) and (c) respectively. The double vector bundle in (b) arises in Poisson geometry: there is a canonical diffeomorphism $T^*(E^*) \to T^*(E)$ and if E (say) has a Lie algebroid structure, then E^* has a Lie-Poisson structure and $T^*(E^*) \to E^*$ is a Lie algebroid.

In a general double vector bundle D, as on the right, the manifold D has two vector bundle structures, one with base A and one with base B (subject to compatibility conditions). Each structure has its dualization operation; let us call them X and Y. It turns out that $XYX = YXY$, up to canonical isomorphism.

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

Taking the dual of a (finite rank) vector bundle is reflexive: the dual of the dual is canonically isomorphic to the original vector bundle, and one may say that duality for vector bundles 'has group C_2'. In particular, in a double vector bundle $X^2 = I$ and $Y^2 = I$, and together with $XYX = YXY$, this shows that the duality of double vector bundles 'has group S_3.'

The lectures will describe these processes and will sketch the triple and 4-fold cases, where new phenomena arise.

References

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