METRIC GEOMETRY OF CARNOT-CARATHÉODORY SPACES WITH C^1-SMOOTH VECTOR FIELDS

Sergey Vodopyanov

Białowieża, Poland,

XXXI Workshop on Geometric Methods in Physics

24 June – 30 June, 2012

REFERENCES

1. S. K. Vodopyanov and M. B. Karmanova, “Local approximation theorem on Carnot manifolds under minimal smoothness”, *Dokl. AN*, 427, No. 6, 731–736 (2009).

2. M. Karmanova and S. Vodopyanov, “Geometry of Carnot–Carathéodory spaces, differentiability and coarea formula”, *Analysis and Mathematical Physics*, Birkhäuser, 284–387 (2009).

3. M. Karmanova “A new approach to investigation of Carnot–Carathéodory geometry", GAFA **21**, no. 6 (2011), 1358–1374.

4. A. V. Greshnov, “A proof of Gromov Theorem on homogeneous nilpotent approximation for C^1-smooth vector fields”, *Mathematicheskie Trudy*, 15, No. 2 (2012).

5. S. G. Basalaev and S. K. Vodopyanov, “Approximate differentiability of mappings of Carnot–Carathéodory spaces”, *Eurasian Math. J.*, 3 (2012): arXiv:1206.5197.

• Mathematical foundation of thermodynamics

• Carnot, Joules There exist thermodynamic states A, B that can not be connected to each other by "adiabatic process".

This impossibility is related to the impossibility of perpetual motion machines.

• Mathematical foundation of thermodynamics

• Carnot, Joules There exist thermodynamic states A, B that can not be connected to each other by "adiabatic process".

This impossibility is related to the impossibility of perpetual motion machines.

• 1909, Carathéodory in order to prove the existence of entropy derived the following statement:

Let $\mathbb{M}$ *be a connected manifold endowed with a corank one distribution. If there exist two points that can not be connected by a horizontal path then the distribution is integrable.*

• **Development**

• Carathéodory 1909, Rashevskiy 1938, Chow 1939: arbitrary two points of $\mathbb{M}$ can be joined by a "horizontal" curve.

• Development

• Carathéodory 1909, Rashevskiy 1938, Chow 1939: arbitrary two points of $\mathbb{M}$ can be joined by a "horizontal" curve.

It follows that $(\mathbb{M}, d_c)$ is a metric space with the subriemannian distance

$$d_c(u,v) = \text{inf}\{L(\gamma) \mid \gamma \text{ is horizontal, } \gamma(0) = u, \gamma(1) = v\}$$

not comparable to Riemannian one.

- **Hörmander, 1967: Hypoelliptic equations**

A problem: *when a distribution solution f to the equation*

$$(X_1^2 + \ldots + X_{n-1}^2 - X_n)f = \varphi \in C^\infty$$

is a smooth function?

Here $X_i \in C^\infty$.

• Hörmander, 1967: Hypoelliptic equations

A problem: *when a distribution solution f to the equation*

$$(X_1^2 + \ldots + X_{n-1}^2 - X_n)f = \varphi \in C^\infty$$

is a smooth function?

Here $X_i \in C^\infty$.

• **Particular case: Kolmogorov's equations**

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} = f$$

• physics (diffusion process), economics (arbitrage theory, some stochastic volatility models of European options), etc.

Hypoelliptic Equations

- **Hörmander (1967)**: sufficient conditions on fields $X_1, \ldots, X_n$:

There exists $M < \infty$ *such that*

- Lie$\{X_1, X_2, \ldots, X_n\} = \text{span}\{X_I(v) \mid |I| \leq M\} = T_v\mathbb{M}$ *for all* $v \in \mathbb{M}$

where

$$X_I(v) = \text{span}\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots](v) : X_{i_j} \in H_1\}$$

for $I = (i_1, i_2, \ldots, i_k)$.

Hypoelliptic Equations

- **Hörmander (1967)**: sufficient conditions on fields $X_1, \ldots, X_n$:

There exists $M < \infty$ *such that*

- Lie$\{X_1, X_2, \ldots, X_n\} = \text{span}\{X_I(v) \mid |I| \leq M\} = T_v\mathbb{M}$ *for all* $v \in \mathbb{M}$ *where*

$$X_I(v) = \text{span}\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots](v) : X_{i_j} \in H_1\}$$

for $I = (i_1, i_2, \ldots, i_k)$.

- M *is the depth of the sub-Riemannian space* $\mathbb{M}$.

Hypoelliptic Equations

• **Hörmander (1967)**: sufficient conditions on fields $X_1, \ldots, X_n$:

There exists $M < \infty$ such that

• Lie$\{X_1, X_2, \ldots, X_n\} = \text{span}\{X_I(v) \mid |I| \leq M\} = T_v\mathbb{M}$ *for all* $v \in \mathbb{M}$ *where*

$$X_I(v) = \text{span}\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots](v) : X_{i_j} \in H_1\}$$

for $I = (i_1, i_2, \ldots, i_k)$.

• M *is called the depth of the sub-Riemannian space* $\mathbb{M}$.

• **Stein (1971)**: The program of studying of geometry of Hörmander vector fields;
description of singularities of fundamental solutions

Geometric control theory

$\diamond$ The linear system of ODE ($x \in \mathbb{M}^N$, $m < N$)

$$\dot{x} = \sum_{i=1}^{n} a_i(t) X_i(x), \qquad X_i \in C^\infty. \tag{1}$$

Geometric control theory

$\diamond$ The linear system of ODE ($x \in \mathbb{M}^N$, $n < N$)

$$\dot{x} = \sum_{i=1}^{n} a_i(t) X_i(x), \qquad X_i \in C^\infty. \tag{2}$$

• Problem: To find bounded measurable functions $a_i(t)$ such that system (5) has a solution with the initial data $x(0) = p$, $x(1) = q$.

Geometric control theory

$\diamond$ The linear system of ODE ($x \in \mathbb{M}^N$, $m < N$)

$$\dot{x} = \sum_{i=1}^{n} a_i(t) X_i(x), \qquad X_i \in C^\infty. \tag{3}$$

• Problem: To find measurable functions $a_i(t)$ such that system (5) has a solution with the initial data $x(0) = p$, $x(1) = q$.

If system (5) has a solution for every $q \in U(p)$ then it is called *locally controllable.*

Geometric control theory

$\diamond$ The linear system of ODE ($x \in \mathbb{M}^N$, $n < N$)

$$\dot{x} = \sum_{i=1}^{n} a_i(t) X_i(x), \qquad X_i \in C^\infty. \tag{4}$$

• Problem: To find measurable functions $a_i(t)$ such that system (5) has a solution with the initial data $x(0) = p$, $x(1) = q$.

If system (5) has a solution for every $q \in U(p)$ then it is called *locally controllable.*

• It is locally controllable iff Lie$\{X_1, X_2, \dots, X_n\} = T\mathbb{M}$, i.e. the "horizontal" distribution $H\mathbb{M} = \{X_1, X_2, \dots, X_n\}$ is bracket-generating.

APPLICATIONS of SUBRIEMANNIAN GEOMETRY

- Thermodynamics
- Non-holonomic mechanics
- Geometric Control Theory
- Subelliptic equation
- Geometric measure theory
- Quasiconformal analysis
- Analysis on metric spaces
- Contact geometry
- Complex variable
- Economics
- Transport problem
- Quantum control
- Neurobiology
- Tomography
- Robotecnics

Carnot–Carathéodory space (C^1-smooth vector fields)

- $\mathbb{M}$ is a connected C^∞-smooth manifold with $\dim_{\text{top}}(\mathbb{M}) = N$;

Carnot–Carathéodory space (C^1-smooth vector fields)

- $\mathbb{M}$ is a connected C^∞-smooth manifold with $\dim_{\mathrm{top}}(\mathbb{M}) = N$;
- in $T\mathbb{M}$ there exists a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M};$$

Carnot–Carathéodory space (C^1-smooth vector fields)

- $\mathbb{M}$ is a connected C^∞-smooth manifold with $\dim_{\text{top}}(\mathbb{M}) = N$;
- in $T\mathbb{M}$ there exists a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M};$$

- $\forall v \in \mathbb{M}\ \exists U(v)$ and vector fields $X_1, X_2, \ldots, X_N \in C^1$ such that

$H_i\mathbb{M}(v) = \text{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;

Carnot–Carathéodory space (C^1-smooth vector fields)

- $\mathbb{M}$ is a connected C^∞-smooth manifold with $\dim_{\text{top}}(\mathbb{M}) = N$;
- in $T\mathbb{M}$ there exists a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M};$$

- $\forall v \in \mathbb{M}\ \exists U(v)$ with vector fields $X_1, X_2, \ldots, X_N \in C^1$ such that

$H_i\mathbb{M}(v) = \text{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;

- $[H_i, H_j] \subset H_{i+j}$, $i, j = 1, \ldots, M-1$. It is equivalent to

$$[X_i, X_j](v) = \sum_{k:\, \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v)$$

where $\deg X_k = \min\{m :\ X_k \in H_m\}$;

Carnot–Carathéodory space (C^1-smooth vector fields)

- $\mathbb{M}$ is a connected C^∞-smooth manifold with $\dim_{\text{top}}(\mathbb{M}) = N$
- in $T\mathbb{M}$ there exists a filtration by subbundles

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

- $\forall v \in \mathbb{M}\ \exists U(v)$ with vector fields $X_1, X_2, \ldots, X_N \in C^1$ such that

$H_i\mathbb{M}(v) = \text{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$, $\dim H_i\mathbb{M}(v) = \dim H_i$;

- $[H_i, H_j] \subset H_{i+j}$, $i, j = 1, \ldots, M-1$;

$\diamond$ *If* $H_{j+1} = \text{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \ldots, [H_k, H_{j+1-k}]\}$ where $k = \lfloor \frac{j+1}{2} \rfloor$, $j = 1, \ldots, M-1$, then $\mathbb{M}$ is called the Carnot manifold.

Classical example.

$\mathbb{M}$ is connected smooth manifold, $\dim \mathbb{M} = N$

$T\mathbb{M}$ is a tangent bundle;

"horizontal" subbundle is

$H\mathbb{M} = \mathrm{span}\{X_1, \ldots, X_n\} \subseteq T\mathbb{M}$ ($n < N$, $X_i \in C^\infty$)

Classical example.

$\mathbb{M}$ is connected smooth manifold, $\dim \mathbb{M} = N$

$T\mathbb{M}$ is a tangent bundle;

"horizontal" subbundle is

$H\mathbb{M} = \text{span}\{X_1, \ldots, X_n\} \subseteq T\mathbb{M}$ ($n < N$, $X_i \in C^\infty$)

There is a filtration $H\mathbb{M} = H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M = T\mathbb{M}$ such that

$$[H_1, H_i] = H_{i+1}, \quad \dim H_i = \text{const}$$

$\Longrightarrow (\mathbb{M}, H\mathbb{M}, \langle \cdot, \cdot \rangle_{H\mathbb{M}})$ defines a subriemannian geometry

M is a depth of the subriemannian space $\mathbb{M}$

Classical example.

$\mathbb{M}$ is connected smooth manifold, $\dim \mathbb{M} = N$

$T\mathbb{M}$ is a tangent bundle;

"horizontal" subbundle is

$H\mathbb{M} = \text{span}\{X_1, \dots, X_n\} \subseteq T\mathbb{M}$ ($n < N$, $X_i \in C^\infty$)

There is a filtration $H\mathbb{M} = H_1 \subseteq H_2 \subseteq \dots \subseteq H_M = T\mathbb{M}$ such that

$$[H_1, H_i] = H_{i+1}, \quad \dim H_i = \text{const}$$

$\Longrightarrow (\mathbb{M}, H\mathbb{M}, \langle \cdot, \cdot \rangle_{H\mathbb{M}})$ defines a subriemannian geometry

M is a depth of the subriemannian space $\mathbb{M}$

• Sub-Riemannian geometry describes changing of physical location when the movement is possible in some prescribed directions.

Examples

1. Heisenberg group $\mathbb{H}^n$

$$\mathbb{M} = \mathbb{R}^{2n+1} : \; X_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t}, \;\; X_{n+i} = \frac{\partial}{\partial x_i} - \frac{x_i}{2}\frac{\partial}{\partial t}, \;\; X_{2n+1} = \frac{\partial}{\partial t}$$

$$H_1 = \mathrm{span}\{X_1, X_2, \ldots, X_{2n}\}, \;\; H_2 = [H_1, H_1] = \mathrm{span}\{X_{2n+1}\}$$

Examples

1. Heisenberg group $\mathbb{H}^n$

$\mathbb{M} = \mathbb{R}^{2n+1}: X_i = \frac{\partial}{\partial x_i} - \frac{x_{n+i}}{2}\frac{\partial}{\partial t},\ X_{n+i} = \frac{\partial}{\partial x_i} - \frac{x_i}{2}\frac{\partial}{\partial t},\ X_{2n+1} = \frac{\partial}{\partial t}$

$H_1 = \mathrm{span}\{X_1, X_2, \dots, X_{2n}\},\ H_2 = [H_1, H_1] = \mathrm{span}\{X_{2n+1}\}$

2. Carnot group is a connected simply connected group Lie G with stratified Lie algebra V:

$$V = V_1 \bigoplus V_2 \bigoplus \dots \bigoplus V_M;\ [V_1, V_i] = V_{i+1}$$

! A Carnot group is a tangent cone to a subriemannian space in a regular point (Mitchell 1985; Gromov, Bellaiche 1996)

Main classical results (proved for smooh enough vector fields)

- 1938–1939 Rashevskiy–Chow theorem;

Main classical results (proved for smooh enough vector fields)

- 1938–1939 Rashevskiy–Chow theorem;
- 1982–1986 Mitchell-Gershkovich-Nagel-Stein-Wainger:

Ball–Box theorem (a ball in the Carnot-Carathéodory metric looks like a box);

Main classical results (proved for smooh enough vector fields)

- 1909–1938–1939 Carathéodory–Rashevskiy–Chow theorem;

- 1982–1986 Mitchell-Gershkovich-Nagel-Stein-Wainger: Ball–Box theorem (a ball in the Carnot-Carathéodory metric looks like a box);

- 1986–1996 Gromov–Mitchell theorem on convergence of rescaled CC-spaces with respect to a fixed point to a nilpotent tangent cone;

Main classical results (proved for smooh enough vector fields)

- 1909–1938–1939 Carathéodory–Rashevskiy–Chow theorem;

- 1982–1986 Mitchell-Gershkovich-Nagel-Stein-Wainger: Ball–Box theorem (a ball in the Carnot-Carathéodory metric looks like a box);

- 1986–1996 Gromov–Mitchell theorem on convergence of rescaled CC-spaces with respect to a fixed point to a nilpotent tangent cone;

- 1996 Gromov theorem on convergence of rescaled vector fields to *nilpotentized* vector fields constituting a basis of graded nilpotent group;

Main classical results (proved for smooh enough vector fields)

- 1909–1938–1939 Carathéodory–Rashevskiy–Chow theorem;

- 1982–1986 Mitchell-Gershkovich-Nagel-Stein-Wainger: Ball–Box theorem (a ball in the Carnot-Carathéodory metric looks like a box);

- 1986–1996 Gromov–Mitchell theorem on convergence of rescaled CC-spaces with respect to a fixed point to a nilpotent tangent cone;

- 1996 Gromov theorem on convergence of rescaled vector fields to *nilpotentized* vector fields constituting a basis of graded nilpotent group;

- 1996 M. Gromov, A. Bellaïche approximation theorem on local behavior of metrics in the given space and in a local tangent cone;

Basic Concepts

Exponential mapping: $u \in \mathbb{M}$, $(v_1, \ldots, v_N) \in \mathbb{R}^N$,

$$\begin{cases} \dot{\gamma}(t) = \sum_{i=1}^{N} v_i X_i(\gamma(t)), & t \in [0,1], \\ \gamma(0) = u. \end{cases}$$

Then $\exp\Big(\sum_{i=1}^{N} v_i X_i\Big)(u) = \gamma(1)$. For each point u, define $\theta_u : U(0) \to \mathbb{M}$ as $\theta_u(v_1, \ldots, v_N) = \exp\Big(\sum_{i=1}^{N} v_i X_i\Big)(u)$.

Basic Concepts

Exponential mapping: $u \in \mathbb{M}$, $(v_1, \dots, v_N) \in \mathbb{R}^N$,

$$\begin{cases} \dot\gamma(t) = \sum\limits_{i=1}^{N} v_i X_i(\gamma(t)), \quad t \in [0,1], \\ \gamma(0) = u. \end{cases}$$

Then $\exp\Bigl(\sum\limits_{i=1}^{N} v_i X_i\Bigr)(u) = \gamma(1)$. For each point u, define $\theta_u : U(0) \to \mathbb{M}$ as $\theta_u(v_1, \dots, v_N) = \exp\Bigl(\sum\limits_{i=1}^{N} v_i X_i\Bigr)(u)$.

Dilatations Δ_τ^u: if $u \in \mathbb{M}$ и $v = \exp\Bigl(\sum\limits_{i=1}^{N} v_i X_i\Bigr)(u)$ then

$$\Delta_\tau^u(v) = \exp\Bigl(\sum_{i=1}^{N} v_i \tau^{\deg X_i} X_i\Bigr)(u)$$

The New Approach to regular CC-spaces: a Local Lie Group at $u \in \mathbb{M}$ for C^1-Smooth Case

$$[X_i, X_j](v) = \sum_{k:\, \deg X_k \le \deg X_i + \deg X_j} c_{ijk}(v) X_k(v).$$

Theorem 1 (2009). Coefficients $\{c_{ijk}(u)\}_{\deg X_k = \deg X_i + \deg X_j} = \{\bar{c}_{ijk}\}$ satisfy Jacobi identity:

$$\sum_k \bar{c}_{ijk}(u)\bar{c}_{kml}(u) + \sum_k \bar{c}_{mik}(u)\bar{c}_{kjl}(u) + \sum_k \bar{c}_{jmk}(u)\bar{c}_{kil}(u) = 0$$

for all $i, j, m, l = 1, \dots, N$, and

$\bar{c}_{ijk} = -\bar{c}_{jik}$ for all $i, j, k = 1, \dots, N$.

Then the collection $\{\bar{c}_{ijk}\}$ defines nilpotent graded Lie algebra.

The New Approach to regular CC-spaces: a Local Lie Group at $u \in \mathbb{M}$ for C^1-Smooth Case

According to the second Lie theorem we take *basis vector fields* $\{(\widehat{X}_i^u)'\}_{i=1}^N$ *in* $\mathbb{R}^N$ *constituting a Lie algebra in such a way that*

$$[(\widehat{X}_i^u)', (\widehat{X}_j^u)'](v) = \sum_{k:\, \text{deg}\, X_k = \text{deg}\, X_i + \text{deg}\, X_j} \bar{c}_{ijk} (\widehat{X}_k^u)'(v),$$

$(\widehat{X}_i^u)' = e_i$, $i = 1, \ldots, N$,

and $\exp = \mathrm{Id}$.

The corresponding Lie group is nilpotent graded Lie group $\mathbb{G}_u\mathbb{M}$

A Local Lie Group $\mathcal{G}^u\mathbb{M}$

In a neighborhod $\mathcal{G}_u \subset \mathbb{M}$ of u push-forwarded vector fields

$\widehat{X}_i^u = D\theta_u(\widehat{X}_i^u)'$ define a structure of local Lie group

in such a way that

$$\theta_u : \mathbb{G}_u\mathbb{M} \to \mathcal{G}_u\mathbb{M}$$

is a local isomorphism of Lie groups.

A Local Lie Group $\mathcal{G}^u\mathbb{M}$

In a neighborhod $\mathcal{G} \subset \mathbb{M}$ of u push-forwarded vector fields

$\widehat{X}_i^u = D\theta_u(\widehat{X}_i^u)'$ they a structure of local Lie group

in such a way that

$$\theta_u : \mathbb{G}_u\mathbb{M} \to \mathcal{G}^u\mathbb{M}$$

is a local isomorphism of Lie groups.

- vector fields $\widehat{X}_i^u$ are left-invariant

Then $(\mathcal{G}, \widehat{X}_1^u, \ldots, \widehat{X}_N^u, \cdot) = \mathcal{G}^u\mathbb{M}$ **is a local Lie group**

- In the case of Carnot manifolds it is called **the local Carnot group**

Quasimetric

Let $v = \exp\Big(\sum_{i=1}^{N} v_i \widehat{X}_i^u\Big)(w)$. Then

$$d_\infty^u(v,w) = \max_{i=1,\dots,N}\{|v_i|^{\frac{1}{\deg X_i}}\}$$

- $d_\infty^u(v,w) \geq 0$; $d_\infty^u(v,w) = 0 \Leftrightarrow v = w$

- $d_\infty^u(v,w) = d_\infty^u(w,v)$

- generalized triangle inequality: for a neighborhood $U \Subset \mathbb{M}$, there exists a constant $c = c(U)$ such that for any $v, s, w \in U$ we have

$$d_\infty^u(v,w) \leq c(d_\infty^u(v,s) + d_\infty^u(s,w))$$

Quasimetric

- d_∞ is defined similarly (with X_i instead of $\widehat{X}_i^u$, $i = 1, \ldots, N$): if $v = \exp\left(\sum_{i=1}^{N} v_i X_i\right)(w)$ then

$$d_\infty(v, w) = \max_{i=1,\ldots,N} \{|v_i|^{\frac{1}{\deg X_i}}\}.$$

- $d_\infty(v, w) \geq 0$; $d_\infty(v, w) = 0 \Leftrightarrow v = w$.

- $d_\infty(v, w) = d_\infty(w, v)$.

- generalized triangle inequality: Do we have locally

$$d_\infty(v, w) \leq c(d_\infty(v, s) + d_\infty(s, w)) \quad \text{for some constant } c?$$

Gromov type nilpotentization theorem

Theorem 2 (2012). For $x \in \mathsf{Box}(g, r_g)$ consider

$$X_i^{\varepsilon}(x) = (\Delta^g_{\varepsilon^{-1}})_* \varepsilon^{\deg X_i} X_i(\Delta^g_{\varepsilon} x), \quad i = 1, \dots, N.$$

Then the following expansion holds:

$$X_i^{\varepsilon}(x) = \widehat{X}_i^g(x) + \sum_{j=1}^{N} a_{ij}(x) \widehat{X}_j^g(x)$$

where $a_{ij}(x) = o(\varepsilon^{\max\{0, \deg X_j - \deg X_i\}})$ for $x \in \mathsf{Box}(g, \varepsilon r_g)$ and $o(\cdot)$ is uniform in g belonging to some compact set of $\mathbb{M}$ as $\varepsilon \to 0$.

Gromov type nilpotentizaton theorem

Theorem 2 (2012). For $x \in \text{Box}(g, r_g)$ consider

$$X_i^\varepsilon(x) = (\Delta^g_{\varepsilon^{-1}})_* \varepsilon^{\deg X_i} X_i(\Delta^g_\varepsilon x), \quad i = 1, \dots, N.$$

Then the following expansion holds:

$$X_i^\varepsilon(x) = \widehat{X}_i^g(x) + \sum_{j=1}^{N} a_{ij}(x) \widehat{X}_j^g(x)$$

where $a_{ij}(x) = o(\varepsilon^{\max\{0, \deg X_j - \deg X_i\}})$ for $x \in \text{Box}(g, \varepsilon r_g)$ and $o(\cdot)$ is uniform in g belonging to some compact set of $\mathbb{M}$ as $\varepsilon \to 0$.

Corollary 1 (**Gromov Type Theorem**): *We have* $X_i^\varepsilon \to \widehat{X}_i^g$ *as* $\varepsilon \to 0$, $i = 1, \dots, N$, *at the points of* $\text{Box}(g, r_g)$ *and this convergence is uniform in* g *belonging to some compact neighborhood.*

Gromov type nilpotentizaton theorem

Theorem 2 (2012). For $x \in \mathrm{Box}(g, r_g)$ consider

$$X_i^\varepsilon(x) = (\Delta^g_{\varepsilon^{-1}})_* \varepsilon^{\deg X_i} X_i(\Delta^g_\varepsilon x), \quad i = 1, \dots, N.$$

Then the following expansion holds:

$$X_i^\varepsilon(x) = \widehat{X}_i^g(x) + \sum_{j=1}^N a_{ij}(x) \widehat{X}_j^g(x)$$

where $a_{ij}(x) = o(\varepsilon^{\max\{0, \deg X_j - \deg X_i\}})$ for $x \in \mathrm{Box}(g, \varepsilon r_g)$ and $o(\cdot)$ is uniform in g belonging to some compact set of $\mathbb{M}$ as $\varepsilon \to 0$.

Corollary 1 (**Gromov Type Theorem**): *We have* $X_i^\varepsilon \to \widehat{X}_i^g$ *as* $\varepsilon \to 0$, $i = 1, \dots, N$, *at the points of* $\mathrm{Box}(g, r_g)$ *and this convergence is uniform in* g *belonging to some compact neighborhood.*

Corollary 2. Generalized triangle inequality holds locally for some constant c: $d_\infty(v, w) \leq c(d_\infty(v, s) + d_\infty(s, w))$.

MAIN RESULT: Comparison of Local Geometries

Let $\mathcal{U} \subset \mathbb{M}$ where $\mathbb{M} \in C^1$:

- $\theta_v(B(0, r_v)) \supset \mathcal{U}$ for all $v \in \mathcal{U}$,
- $\mathcal{G}^u\mathbb{M} \supset \mathcal{U}$ for all $u \in \mathcal{U}$,
- $\theta_v^u(B(0, r_{u,v})) \supset \mathcal{U}$ for all $u, v \in \mathcal{U}$.

Theorem 3 (2009). *Let $u, u', v \in \mathcal{U} \Subset \mathbb{M}$. Assume that $d_\infty(u, u') = O(\varepsilon)$ and $d_\infty(u, v) = O(\varepsilon)$, and consider points*

$$w_\varepsilon = \exp\Big(\sum_{i=1}^{N} w_i \varepsilon^{\deg X_i} \widehat{X}_i^u\Big)(v) \text{ and } w'_\varepsilon = \exp\Big(\sum_{i=1}^{N} w_i \varepsilon^{\deg X_i} \widehat{X}_i^{u'}\Big)(v).$$

Then

$$\max\{d_\infty^u(w_\varepsilon, w'_\varepsilon), d_\infty^{u'}(w_\varepsilon, w'_\varepsilon)\} = o(\varepsilon)$$

where $o(\varepsilon)$ is uniform in $u, u', v \in \mathcal{U}$.

Corollaries

4) Local Approximation Theorem for d_∞-quasimetric (2009):

Let $v, w \in \mathrm{Box}(g, \varepsilon) \subset \mathbb{M}$. Then

$$|d^u_\infty(v, w) - d_\infty(v, w)| = o(\varepsilon).$$

Corollaries

Assumption: Suppose that $\mathbb{M}$ is a Carnot manifold.

5) Rashevsky–Chow type Theorem (2012): *Any two points $x, y \in \mathbb{M}$ can be connected by a horizontal curve γ (i. e., $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0, 1]$).*

The intrinsic metric on Carnot–Carathéodory space

$$d_c(u, v) = \inf_{\substack{\gamma \text{ is horizontal} \\ \gamma(0)=u, \gamma(1)=v}} \{L(\gamma)\}$$

Corollaries

Assumption: Suppose that $\mathbb{M}$ is a Carnot manifold.

5) Rashevsky–Chow type Theorem (2012): *Any two points $x, y \in \mathbb{M}$ can be connected by a horizontal curve γ (i. e., $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0,1]$).*

The intrinsic metric on Carnot–Carathéodory space

$$d_{cc}(u,v) = \inf_{\substack{\gamma \text{ is horizontal} \\ \gamma(0)=u, \gamma(1)=v}} \{L(\gamma)\}$$

6) Local Approximation Theorem for d_{cc}-metric (2009): *For $v, w \in B_{cc}(u,\varepsilon)$, we have*

$$|d_{cc}(v,w) - d^u_{cc}(v,w)| = o(\varepsilon).$$

Corollaries

7) Mitchell-Gershkovich-Nagel-Stein-Wainger theorem type Ball–Box Theorem (2012). *For* $\mathcal{U} \Subset \mathbb{M}$*, there exist constants* $c(\mathcal{U}) \leq C(\mathcal{U})$ *such that*

$$c(\mathcal{U})d_\infty(x,y) \leq d_{cc}(x,y) \leq C(\mathcal{U})d_\infty(x,y),$$

where $x, y \in \mathcal{U}$*, and* $d_{cc}(x,y)$ *is a Carnot–Carathéodory metric.*

Corollaries

7) Mitchell-Gershkovich-Nagel-Stein-Wainger theorem type Ball–Box Theorem (2011). *For* $\mathcal{U} \Subset \mathbb{M}$, *there exist constants* $c(\mathcal{U}) \le C(\mathcal{U})$ *such that*

$$c(\mathcal{U})d_\infty(x,y) \le d_{cc}(x,y) \le C(\mathcal{U})d_\infty(x,y),$$

where $x, y \in \mathcal{U}$, *and* $d_{cc}(x,y)$ *is a Carnot–Carathéodory metric.*

Proof: $d^u_{cc}(u,w)(1-o(1)) \le d_{cc}(u,w) \le d^u_{cc}(u,w)(1+o(1))$;

$d^u_\infty(u,w)(1-o(1)) \le d_\infty(u,w) \le d^u_\infty(u,w)(1+o(1))$;

$d^u_{cc}(u,w) \sim d^u_\infty(u,w)$.

Application to Geometric control theory

$\diamond$ The linear system of ODE ($x \in \mathbb{M}^N$, $n < N$)

$$\dot{x} = \sum_{i=1}^{n} a_i(t) X_i(x), \quad X_i \in C^1. \tag{5}$$

• Problem: To find measurable functions $a_i(t)$ such that system (5) has a solution with the initial data $x(0) = p$, $x(1) = q$.

If system (5) has a solution for every $q \in U(p)$ then it is called *locally controllable.*

• (5) locally controllable if "horizontal" v.f. $\{X_1, \dots, X_n\}$ can be extended to the system of v.f. constituting a structure of a Carnot manifold.

Main Applications

• *sub-Riemannian differentiability theory*: Rademacher-type and Stepanov-type Theorems on sub-Riemannian differentiability of mappings of Carnot manifolds (S. Vodopyanov)

• *geometric measure theory on sub-Riemannian structures*: area formula for intrinsically Lipschitz mappings of Carnot manifolds, coarea formula for C^{M+1}-smooth mappings of Carnot manifolds (M. Karmanova; S. Vodopyanov)

• *geometry of non-equiregular Carnot–Carathéodory spaces* (S. Selivanova)

Sub-Riemannian Differentiability [V2007]

Definition. A mapping $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is hc-differentiable* at $u \in \mathbb{M}$ if there exists a horizontal homomorphism

$$L_u : (\mathcal{G}^u, d_{cc}^u) \to (\mathcal{G}^{\varphi(u)}, d_{cc}^{\varphi(u)})$$

of local Carnot groups such that

$$\tilde{d}_{cc}(\varphi(w), L_u(w)) = o(d_{cc}(u, w)),\ \ E \cap \mathcal{G}^u \ni w \to u.$$

- For mappings of Carnot groups, this notion coincides with the definition of $\mathcal{P}$-differentiability [Pansu]

- Denote the hc-differential of φ at u by the symbol $\widehat{D}\varphi(u)$

Sub-Riemannian Differentiability [V2007]

Rademacher-Type Theorem. *Suppose that a mapping* $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is Lipschitz with respect to* d_∞ *и* $\tilde{d}_\infty$. *Then* φ *is* hc*-differentiable almost everywhere.*

Stepanov-Type Theorem. *Suppose that a mapping* $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is such that*

$$\overline{\lim_{y\to x}} \frac{\tilde{d}_{cc}(\varphi(y), \varphi(x))}{d_{cc}(y, x)}$$

almost everywhere. Then φ *is* hc*-differentiable almost everywhere.*

Theorem. *Suppose that* $\varphi : (\mathbb{M}, d_{cc}) \to (\widetilde{\mathbb{M}}, \tilde{d}_{cc})$ *is* C^1_H*-smooth and contact (i. e.,* $D_H\varphi[H\mathbb{M}] \subset H\widetilde{\mathbb{M}}$*). Then* φ *is continuously* hc*-differentiable everywhere.*

Sub-Riemannian Area Formula [K2008]

- the sub-Riemannian Jacobian

$$\mathcal{J}^{SR}(\varphi, y) = \sqrt{\det(\widehat{D}\varphi(y)^*\widehat{D}\varphi(y))}.$$

Theorem. *Let* $\varphi : \mathbb{M} \to \widetilde{\mathbb{M}}$ *be a Lipschitz with respect to* d_{cc} *and* $\tilde{d}_{cc}$ *mapping of Carnot manifolds. Then, the area formula holds:*

$$\int\limits_{\mathbb{M}} f(y)\mathcal{J}^{SR}(\varphi, y)\, d\mathcal{H}^{\nu}(y) = \int\limits_{\widetilde{\mathbb{M}}} \sum_{y:\, y\in\varphi^{-1}(x)} f(y)\, d\mathcal{H}^{\nu}(x),$$

where $f : \mathbb{M} \to \mathbb{E}$ ($\mathbb{E}$ *is an arbitrary Banach space*) *is such that the function* $f(y)\sqrt{\det(\widehat{D}\varphi(y)^*\widehat{D}\varphi(y))}$ *is integrable. Here Hausdorff measures are constructed with respect to quasimetrics* d_2 *(in the preimage) and* $\tilde{d}_2$ *(in the image) with the normalizing factor* ω_ν.

Sub-Riemannian Coarea Formula [KV2009]

- the sub-Riemannian coarea factor

$$\mathcal{J}^{SR}_{\widetilde{N}}(\varphi,x) = \sqrt{\det(\widehat{D}\varphi(x)\widehat{D}\varphi(x)^*)} \cdot \frac{\omega_N}{\omega_\nu}\frac{\omega_{\widetilde{\nu}}}{\omega_{\widetilde{N}}}\frac{\omega_{\nu-\widetilde{\nu}}}{\prod\limits_{k=1}^{M}\omega_{n_k-\tilde{n}_k}}.$$

Theorem. *Suppose that* $\varphi \in C^{M+1}(\mathbb{M},\widetilde{\mathbb{M}})$ *is a contact mapping of two Carnot manifolds,* $\dim H_1\mathbb{M} \geq \dim \widetilde{H}_1\widetilde{\mathbb{M}}$, $\dim H_i\mathbb{M} - \dim H_{i-1}\mathbb{M} \geq \dim \widetilde{H}_i\widetilde{\mathbb{M}} - \dim \widetilde{H}_{i-1}\widetilde{\mathbb{M}}$, $i = 2,\ldots,M$. *Then the following coarea formula*

$$\int\limits_{\mathbb{M}} \mathcal{J}^{SR}_{\widetilde{N}}(\varphi,x) f(x)\, d\mathcal{H}^{\nu}(x) = \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\widetilde{\nu}}(z) \int\limits_{\varphi^{-1}(z)} f(u)\, d\mathcal{H}^{\nu-\widetilde{\nu}}(u)$$

holds, where $f:\mathbb{M}\to\mathbb{E}$ *(*$\mathbb{E}$ *is an arbitrary Banach space) is such that the product* $\mathcal{J}^{SR}_{\widetilde{N}}(\varphi,x)f(x):\mathbb{M}\to\mathbb{E}$ *is integrable.*