Homotopy Classification of Super Line Bundles

S. Varsaie

Institute for Advanced Studies in Basic Sciences

June, 2012

OUTLINE

Introduction

Homotopoy classiffication of super line bundles

It has been studied in relation to generalizing the concept of the Chern classes in supergeometry.
The latter problem has been discussed in different papers.

Introduction

Homotopoy classiffication of super line bundles

It has been studied in relation to generalizing the concept of the Chern classes in supergeometry.
The latter problem has been discussed in different papers.

Y. I Manin and A. A. Voronov; Schubert supercells, Funktsional Anal. i Prilozhen. 18(1984)
Y. I.Manin and I. B. Penkov; The formalism of left and right connections, Lecture on Supermanifolds, geometrical methods and conformal groups, 3-13, World Sci. Publishing, 1989.

Chern class

From classifying space viewpoint, in common geometry, Chern classes of a vector bundle $E \to M$ measure to what extent the vector bundle looks like a universal bundle.

Chern class

From classifying space viewpoint, in common geometry, Chern classes of a vector bundle $E \to M$ measure to what extent the vector bundle looks like a universal bundle.

Homotopy classification

For this, one may associate isomorphic class of the vector bundle $E \to M$ to homotopy class of a continuous map from M to a Grassmann manifold.

Chern class

From classifying space viewpoint, in common geometry, Chern classes of a vector bundle $E \to M$ measure to what extent the vector bundle looks like a universal bundle.

Homotopy classification

For this, one may associate isomorphic class of the vector bundle $E \to M$ to homotopy class of a continuous map from M to a Grassmann manifold.

Generalizing the concept of the Chern classes in supergeometry

Via classifying space, one needs a generalization of Grassmannians and a generalization of the homotopy classiffication theorem.

Projective superspaces are not satisfactory

Indeed, it can be shown that projective superspaces and projective spaces are homotopy equivalence.
Thus (Cech-)cohomology of projective superspaces do not contain any information about superstructure.

PROJECTIVE SUPERSPACES ARE NOT SATISFACTORY

Indeed, it can be shown that projective superspaces and projective spaces are homotopy equivalence.
Thus (Cech-)cohomology of projective superspaces do not contain any information about superstructure.

ν- PROJECTIVE SPACES

There is another generalization of projective spaces, called ν- projective spaces, which are useful here.

Some defenitions

Standard ν- domains

By a standard ν- domain, we mean a standard superdomain $(\mathbb{C}^m, \mathcal{O})$ such that $\mathcal{O}$ carries a $\mathbb{C}_\nu$- module structure where $\mathbb{C}_\nu = \mathbb{C}[\nu]$ is a ring generated by ν with relation $\nu^2 = 1$. In addition, for each x, $\nu\mathcal{O}_x^o \subset \mathcal{O}_x^e$ and $\nu\mathcal{O}_x^e \subset \mathcal{O}_x^o$ where $\mathcal{O}^o, \mathcal{O}^e$ are odd and even parts of $\mathcal{O}$ respectively.

SOME DEFENITIONS

STANDARD ν- DOMAINS

By a standard ν- domain, we mean a standard superdomain $(\mathbb{C}^m, \mathcal{O})$ such that $\mathcal{O}$ carries a $\mathbb{C}_\nu$- module structure where $\mathbb{C}_\nu = \mathbb{C}[\nu]$ is a ring generated by ν with relation $\nu^2 = 1$. In addition, for each x, $\nu\mathcal{O}_x^o \subset \mathcal{O}_x^e$ and $\nu\mathcal{O}_x^e \subset \mathcal{O}_x^o$ where $\mathcal{O}^o, \mathcal{O}^e$ are odd and even parts of $\mathcal{O}$ respectively.

MORPHISMS

By a morphism of standard ν- domains, we mean a pair (ϕ, ψ) where (ϕ, ψ) is a morphism of standard superdomains i.e.
i) $\phi : \mathbb{C}^m \to \mathbb{C}^k$ is continuous map.
ii) $\psi : \mathcal{O}_k \to \phi_*\mathcal{O}_m$ is a morphism of sheaves of supercommutative $\mathbb{Z}_2$-graded local rings.
Moreover $\psi : \mathcal{O}_{\phi(x)} \to \mathcal{O}_x$ is a $\mathbb{C}_\nu$- module homomorphism. Then one may write $(\phi, \psi) : (\mathbb{C}^m, \mathcal{O}_m) \to (\mathbb{C}^k, \mathcal{O}_k)$

Even and odd neighbourhoods

By ν- projective space of dimension $m|n$, we mean a supermanifold which is constructed by glueing some copies of standard ν- domain $(\mathbb{C}^m, \mathcal{O})$ where $\mathcal{O} = C^\infty(\mathbb{C}^m) \otimes \wedge\mathbb{C}^n$.

Even and odd neighbourhoods

By ν- projective space of dimension $m|n$, we mean a supermanifold which is constructed by glueing some copies of standard ν- domain $(\mathbb{C}^m, \mathcal{O})$ where $\mathcal{O} = C^\infty(\mathbb{C}^m) \otimes \wedge\mathbb{C}^n$.

For expressing the rules for glueing, one may set (even) neighbourhoods $U_i = \mathbb{C}^m$, $1 \leq i \leq m+1$, and (odd) neighbourhoods $U_j = \mathbb{C}^n$, $1 \leq j \leq n$.

Rules for gluing

Now to each (even) neighbourhood U_i, associate a $1|0 \times (m+1)|n$ matrix in standard format say A_i, such that the i-th entry is one. This matrix consists of even and odd smooth coordinates as follows:

$$A_i = (x_1, ..., x_{i-1}, 1, x_i, ...x_m; e_1, ..., e_n)$$

Rules for gluing

Now to each (even) neighbourhood U_i, associate a $1|0 \times (m+1)|n$ matrix in standard format say A_i, such that the i-th entry is one. This matrix consists of even and odd smooth coordinates as follows:

$$A_i = (x_1, ..., x_{i-1}, 1, x_i, ...x_m; e_1, ..., e_n)$$

To each (odd) neighbourhood U_j, associate a $1|0 \times m|(n+1)$ matrix in standard format say A_j such that the j- th entry is $\nu 1$. This matrix is as follows:

$$A_j = (x_1, ..., x_m; e_1, ..., e_{j-1}, \nu 1, e_j, ..., e_n)$$

Rules for gluing

Now to each (even) neighbourhood U_i, associate a $1|0 \times (m+1)|n$ matrix in standard format say A_i, such that the i-th entry is one. This matrix consists of even and odd smooth coordinates as follows:

$$A_i = (x_1, ..., x_{i-1}, 1, x_i, ...x_m; e_1, ..., e_n)$$

To each (odd) neighbourhood U_j, associate a $1|0 \times m|(n+1)$ matrix in standard format say A_j such that the j- th entry is $\nu 1$. This matrix is as follows:

$$A_j = (x_1, ..., x_m; e_1, ..., e_{j-1}, \nu 1, e_j, ..., e_n)$$

Two (even) neighbourhoods U_{i_1}, U_{i_2} intersect if in A_{i_1} the i_2-th entry say $b^{i_1}_{i_2}$ is invertible. Then the rules for gluing coming from the following equality:

$$A_{i_2} = (b^{i_1}_{i_2})^{-1} A_{i_1}$$

Two neighbourhoods U_i and U_j intersect if $\nu(b^i_j)$ is invertible where b^i_j is the j-th entry in A_i. Then the rules for gluing coming from the following equality:

$$A_j = \nu((\nu(b^i_j))^{-1}\nu(A_i))$$

where

$$\nu(A_i) = (\nu e_1, ..., \nu e_n; \nu x_1, ..., \nu 1, ...\nu x_m)$$

Two neighbourhoods U_i and U_j intersect if $\nu(b^i_j)$ is invertible where b^i_j is the j-th entry in A_i. Then the rules for gluing coming from the following equality:

$$A_j = \nu((\nu(b^i_j))^{-1}\nu(A_i))$$

where

$$\nu(A_i) = (\nu e_1, ..., \nu e_n; \nu x_1, ..., \nu 1, ...\nu x_m)$$

The isomorphisms Φ_{ts}, $1 \leq t, s \leq m + n + 1$, corresponding to rules for gluing on intersection U_t and U_s, satisfy the gluing conditions. Thus the ring spaces $(U_t, \mathcal{O}_t)$ may be glued through the isomorphisms Φ_{ts} to form a ring space ${}^{\nu}\mathcal{P}^{m|n} = (\mathbb{C}P^m, \bar{\mathcal{O}})$ which is called ν- projective space.

Main results

Theorem

1. *There exists a canonical super line bundle of rank* $1|0$ *denoted by* ${}^{\nu}\mathcal{L}$*, over* ${}^{\nu}\mathcal{P}^{m|n}$
2. *there exists a cohomology class denoted by* ${}^{\nu}c$ *in* $H^2(\mathbb{C}P^m, \mathbb{Z}[\nu])$ *associated to the isomorphism class of* ${}^{\nu}\mathcal{L}$*.*
   This class may be considered as super geometric generalization of universal Chern class.
3. *Let* $\mathcal{E}$ *be a superline bundle over* $\mathcal{M} = (M, \mathcal{A})$*. For some pair of natural numbers, say* (m, n)*, there exists a morphism*(ϕ, ψ) *from* $\mathcal{M}$ *to* ${}^{\nu}\mathcal{P}^{m|n}$*.*

ν- CLASS

ν- CLASS

It is seen that $\phi * ({}^{\nu}c)$ is an element of $H^2(M, \mathbb{Z}[\nu])$. It is called ν- class of $\mathcal{E}$. One may consider this element as an analog of the Chern classes of line bundles in supergeometry.

ν- PART

ν- part of this element measures to what extent the even and odd elements of $\mathcal{E}$ are associated to each other.

Conclusion

Higher order ν- classes

Here (first) ν- classes for super line bundles are introduced. It is possible to introduce (higher order) ν- classes for super vector bundles. Some evidences imply that these classes can be identified by geometric invariants.

ν- Geometry

In this regard, a geometry can be developed on ν- manifolds. Since this geometry is developed on base ring $\mathbb{C}[\nu]$, thus it may be called ν-geometry.

Refrences

- V. S. Varadarajan; Supersymmetry for Mathematicians: An Introduction (Courant Lecture Notes: 11), 2004.
- S. Varsaie; ν- Projective spaces, ν- line bundles and Chern classes, Int. J. Geom. Methods Mod. Phys.8(2011), no. 2
- S. Varsaie; ν- Manifolds, Int. J. Geom. Methods Mod. Phys.8(2011), no. 6
- S. Varsaie; ν- Classes, Int. J. Geom. Methods Mod. Phys.9(2012), no. 4