

Cohomology of flag varieties and Bethe algebras

VITALY TARASOV

Department of Mathematical Sciences, IUPUI

and

St. Petersburg Branch of Steklov Mathematical Institute

Joint work with V. Gorbounov, R. Rimányi, A. Varchenko

arXiv:1204.5138

XXXI Workshop on Geometric Methods in Physics
Białowieża, June 24–30, 2012

Equivariant cohomology of the cotangent bundles of partial flag varieties.

Let $\lambda_1, \dots, \lambda_N \in \mathbb{Z}_{\geq 0}$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$, $n = |\boldsymbol{\lambda}| = \sum_{i=1}^N \lambda_i$.

The partial flag variety $\mathcal{F}_{\boldsymbol{\lambda}}$ parametrise partial flags

$$0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{C}^n, \quad \dim F_i / F_{i-1} = \lambda_i.$$

The $GL_n(\mathbb{C}) \times \mathbb{C}^*$ -equivariant cohomology algebra

$$H_{\boldsymbol{\lambda}} = H_{GL_n \times \mathbb{C}^*}^*(T^*\mathcal{F}_{\boldsymbol{\lambda}}; \mathbb{C})$$

has the following description:

$$H_{\boldsymbol{\lambda}} = \mathbb{C}[\Gamma; \mathbf{z}; h]^{S_{\lambda_1} \times \dots \times S_{\lambda_N} \times S_n} / \langle \text{relations} \rangle,$$

where $\Gamma = (\gamma_{i,j})_{i=1, \dots, N, j=1, \dots, \lambda_i}$, $\mathbf{z} = (z_1, \dots, z_n)$, and the relations are

$$p(\gamma_{1,1}, \dots, \gamma_{1,\lambda_1}, \dots, \gamma_{N,1}, \dots, \gamma_{N,\lambda_N}) = p(z_1, \dots, z_n)$$

for every symmetric polynomial p in n variables.

An isomorphism $\zeta_{\lambda} : H_{\lambda} \rightarrow \mathbb{C}[\Gamma; h]^{S_{\lambda_1} \times \dots \times S_{\lambda_N}} = \mathbb{C}[\Gamma; h]^{S_{\lambda}}$,

$$\zeta_{\lambda} : [f(\Gamma; z; h)] \mapsto f(\Gamma; \Gamma; h) \in \mathbb{C}[\Gamma; h]^{S_{\lambda}}.$$

The integration on \mathcal{F}_{λ} and the equivariant integration on $T^*\mathcal{F}_{\lambda}$:

$$\int [f] = \sum_{I \in \mathcal{I}_{\lambda}} \frac{f(z_I; z; h)}{R(z_I)}, \quad \oint [f] = \sum_{I \in \mathcal{I}_{\lambda}} \frac{f(z_I; z; h)}{R(z_I) Q(z_I)}.$$

where $[f] \in H_{\lambda}$ and the rest of the notation is as follows:

\mathcal{I}_{λ} is the collection of sequences $I = (I_1, \dots, I_N)$ with the properties

$$I_j = \{i_{j,1}, \dots, i_{j,\lambda_j}\} \subset \{1, \dots, n\}, \quad \bigcup_{j=1}^N I_j = \{1, \dots, n\};$$

$$z_I = (z_{i_{1,1}}, \dots, z_{i_{1,\lambda_1}}; \dots; z_{i_{N,1}}, \dots, z_{i_{N,\lambda_N}}),$$

$$R(z_I) = \prod_{1 \leq a < b \leq N} \prod_{\substack{i \in I_a \\ j \in I_b}} (z_i - z_j), \quad Q(z_I) = \prod_{1 \leq a < b \leq N} \prod_{\substack{i \in I_a \\ j \in I_b}} (z_i - z_j + h).$$

Let $Q(\Gamma; h) = \prod_{1 \leq a < b \leq N} \prod_{i=1}^{\lambda_a} \prod_{j=1}^{\lambda_b} (\gamma_{a,i} - \gamma_{b,j} + h)$, so that

$$Q(z_I; h) = Q(\mathbf{z}_I). \text{ Then } \int [f] = \int [f Q].$$

The integration on $\mathcal{F}_{\boldsymbol{\lambda}}$ defines a map $\int : H_{\boldsymbol{\lambda}} \rightarrow \mathbb{C}[z; h]^{S_n}$.

The integration on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$ defines a map $\int : H_{\boldsymbol{\lambda}} \rightarrow Z^{-1} \mathbb{C}[z; h]^{S_n}$,

$$\text{where } Z = \prod_{\substack{i,j=1 \\ i \neq j}}^n (z_i - z_j + h).$$

The Poincare pairing on $T^*\mathcal{F}_{\boldsymbol{\lambda}}$:

$$H_{\boldsymbol{\lambda}} \otimes H_{\boldsymbol{\lambda}} \rightarrow Z^{-1} \mathbb{C}[z; h]^{S_n}, \quad f \otimes g \mapsto \langle f, g \rangle = \int f g.$$

Quantum multiplication \circ on H_{λ} is an associative commutative product depending on parameters q_1, \dots, q_N .

Properties:

- a) $f \circ g$ is a power series in $q_2/q_1, \dots, q_N/q_{N-1}$;
- b) $f \circ g = fg + \dots$ as $q_{i+1}/q_i \rightarrow 0$, $i = 1, \dots, N-1$;
- c) $f \circ 1 = f$;
- d) $\int (f \circ g_1) g_2 = \int (f \circ g_2) g_1$, in particular $\int f \circ g = \int fg$;

For $f \in H_{\lambda}$, consider the operator $f \circ : g \mapsto f \circ g$.

- e) The operators $f \circ$ are symmetric with respect to the Poincare pairing on $T^* \mathcal{F}_{\lambda}$;
- f) The connection $\nabla_i^\circ = \kappa q_i \frac{\partial}{\partial q_i} - (\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}) \circ$ is flat for any $\kappa \neq 0$,

$$[\nabla_i^\circ, \nabla_j^\circ] = 0, \quad i, j = 1, \dots, N.$$

Quantum integrable model. Space of states.

$$\mathbb{C}^N = \mathbb{C}v_1 \oplus \dots \oplus \mathbb{C}v_N, \quad V = (\mathbb{C}^N)^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda.$$

$$I = (I_1, \dots, I_N) \in \mathcal{I}_\lambda, \quad |\lambda| = n,$$

$$I_j = \{i_{j,1}, \dots, i_{j,\lambda_j}\} \subset \{1, \dots, n\}, \quad \bigcup_{j=1}^N I_j = \{1, \dots, n\},$$

$$v_I = v_{k_1} \otimes \dots \otimes v_{k_n} \in V, \text{ where } k_i = j \text{ if } i \in I_j.$$

For example, $N = 3$, $n = 6$, $\lambda = (3, 1, 2)$,

$$I = (\{2, 3, 6\}, \{4\}, \{1, 5\}), \quad v_I = v_3 \otimes v_1 \otimes v_1 \otimes v_2 \otimes v_3 \otimes v_1.$$

V_λ — the weight subspace of weight λ — is spanned by v_I , $I \in \mathcal{I}_\lambda$.

$$I_\lambda^{\min} = (\{1, \dots, \lambda_1\}, \dots, \{n - \lambda_N + 1, \dots, n\}),$$

$$I_\lambda^{\max} = (\{n, \dots, n - \lambda_1 + 1\}, \dots, \{\lambda_N, \dots, 1\}).$$

$$\sigma \in S_n, \quad A = \{a_1, \dots, a_k\} \subset \{1, \dots, n\}, \quad \sigma(A) = \{\sigma(a_1), \dots, \sigma(a_k)\}.$$

$$I = (I_1, \dots, I_N) \in \mathcal{I}_\lambda, \quad \sigma(I) = (\sigma(I_1), \dots, \sigma(I_N)) \in \mathcal{I}_\lambda.$$

Simple transpositions $s_1, \dots, s_{n-1} \in S_n$.

Permutation operators $P^{(i,i+1)} \in \text{End}(V)$, $P^{(i,i+1)} v_I = v_{s_i(I)}$.

Operators $\tilde{s}_1, \dots, \tilde{s}_{n-1}$ act on V -valued functions of z_1, \dots, z_n, h ,

$$\tilde{s}_i f(\mathbf{z}; h) = \frac{(z_i - z_{i+1}) P^{(i,i+1)} + h}{z_i - z_{i+1} + h} f(\dots, z_{i+1}, z_i, \dots; h).$$

The assignment $s_i \mapsto \tilde{s}_i$ defines an action of S_n .

Operators $\hat{s}_1, \dots, \hat{s}_{n-1}$ act on functions of z_1, \dots, z_n, h ,

$$\hat{s}_i f(\mathbf{z}; h) = \frac{z_i - z_{i+1} - h}{z_i - z_{i+1}} f(\dots, z_{i+1}, z_i, \dots; h) + \frac{h}{z_i - z_{i+1}} f(\mathbf{z}; h).$$

The assignment $s_i \mapsto \hat{s}_i$ defines an action of S_n .

Let $f(\mathbf{z}; h) = \sum_I f_I(\mathbf{z}; h) v_I$, where f_I are scalar functions.

Then $f = \tilde{\sigma} f$ for all $\sigma \in S_n$ iff $f_{\sigma(I)} = \hat{\sigma} f_I$ for all I and $\sigma \in S_n$.

The space $\frac{1}{D}\mathcal{V}_{\lambda}^-$: $f \in \frac{1}{D}\mathcal{V}_{\lambda}^-$ iff $f = \tilde{\sigma}f$ for all $\sigma \in S_n$ and

$$f \in V_{\lambda} \otimes \frac{1}{D}\mathbb{C}[z; h], \quad D = \prod_{1 \leq i < j \leq n} (z_i - z_j + h).$$

An isomorphism $\vartheta_{\lambda} : \mathbb{C}[\Gamma; h]^{S_{\lambda}} \rightarrow \frac{1}{D}\mathcal{V}_{\lambda}^-$,

$$\vartheta_{\lambda}(f) = \frac{1}{\lambda_1! \dots \lambda_N!} \sum_{\sigma \in S_n} \hat{\sigma}\left(\frac{1}{Q} \ddot{f}\right) v_{\sigma(I_{\lambda}^{\min})},$$

$$\ddot{f}(z; h) = f(z_{I_{\lambda}^{\min}}; h), \quad Q(\Gamma; h) = \prod_{1 \leq a < b \leq N} \prod_{i=1}^{\lambda_a} \prod_{j=1}^{\lambda_b} (\gamma_{a,i} - \gamma_{b,j} + h).$$

Functions $\xi_I \in V_{\lambda} \otimes \frac{1}{D}\mathbb{C}[z; h]$: $\xi_{\sigma(I)} = \tilde{\sigma}\xi_I$ for all I and $\sigma \in S_n$,
and $\xi_{I_{\lambda}^{\max}} = v_{I_{\lambda}^{\max}}$,

$$\vartheta_{\lambda}(f) = \sum_{I \in \mathcal{I}_{\lambda}} \frac{f(z_I; h)}{R(z_I)} \xi_I, \quad R(z_I) = \prod_{1 \leq a < b \leq N} \prod_{\substack{i \in I_a \\ j \in I_b}} (z_i - z_j).$$

Bilinear form $\mathcal{S} : V \otimes V \rightarrow \mathbb{C}$, $\mathcal{S}(v_I, v_J) = \delta_{IJ}$.

$\Pi \in \text{End}(V)$: $\Pi(w_1 \otimes \dots \otimes w_n) = w_n \otimes \dots \otimes w_1$, $w_1, \dots, w_n \in \mathbb{C}^N$.

Bilinear form on V -valued functions of \mathbf{z}, h : $\langle\!\langle f, g \rangle\!\rangle = \mathcal{S}(f, \check{g})$,

where $\check{g}(\mathbf{z}, h) = \Pi(g(z_n, \dots, z_1, h))$.

For $f, g \in \frac{1}{D}\mathcal{V}_{\lambda}^-$, $\langle\!\langle f, g \rangle\!\rangle = \langle\!\langle g, f \rangle\!\rangle \in Z^{-1} \mathbb{C}[\mathbf{z}; h]^{S_n}$, where

$$Z = \prod_{\substack{i,j=1 \\ i \neq j}}^n (z_i - z_j + h), \quad Q(\mathbf{z}_I) = \prod_{1 \leqslant a < b \leqslant N} \prod_{\substack{i \in I_a \\ j \in I_b}} (z_i - z_j + h).$$

For functions ξ_I , $\langle\!\langle \xi_I, \xi_J \rangle\!\rangle = \delta_{IJ} \frac{R(\mathbf{z}_I)}{Q(\mathbf{z}_I)}$.

For $f, g \in \mathbb{C}[\Gamma; h]^{S_{\lambda}}$, $\langle f, g \rangle = \langle\!\langle \vartheta_{\lambda}(f), \vartheta_{\lambda}(g) \rangle\!\rangle$.

Distinguished vectors: $v_{\lambda}^- = \vartheta_{\lambda}(1)$, $v_{\lambda}^+ = \vartheta_{\lambda}(Q) = \sum_{I \in \mathcal{I}_{\lambda}} v_I$.

Gelfand-Zetlin algebra and Bethe algebra.

Label the factors of $\mathbb{C}^N \otimes V = \mathbb{C}^N \otimes (\mathbb{C}^N)^{\otimes n}$ by $0, 1, \dots, n$.

$$L(u) = \frac{u - z_1 + h P^{(0,1)}}{u - z_1} \cdots \frac{u - z_N + h P^{(0,N)}}{u - z_N}$$

is an $N \times N$ matrix with entries $L_{ij}(u)$,

$$L_{ij}(u) = \delta_{ij} + h \sum_{s=1}^{\infty} L_{ij}^{\{s\}} u^{-s}, \quad L_{ij}^{\{s\}} \in \text{End}(V) \otimes \mathbb{C}[z; h].$$

$\mathcal{Y} \subset \text{End}(V) \otimes \mathbb{C}[z; h]$ is the unital subalgebra generated by all $L_{ij}^{\{s\}}$.

\mathcal{Y} acts on $V \otimes \mathbb{C}[z; h]$ and commutes with the operators $\hat{\sigma}$, $\sigma \in S_n$.

Thus \mathcal{Y} acts on $\frac{1}{D} \mathcal{V}^- = \bigoplus_{|\lambda|=n} \frac{1}{D} \mathcal{V}_{\lambda}^-$.

The assignment $\varpi : L_{ij}^{\{s\}} \mapsto L_{ji}^{\{s\}}$ defines an antiautomorphism of \mathcal{Y} .

For any $X \in \mathcal{Y}$ and any $f, g \in \frac{1}{D} \mathcal{V}^-$, $\langle\langle Xf, g \rangle\rangle = \langle\langle f, \varpi(X)g \rangle\rangle$.

For $\mathbf{i} = \{1 \leq i_1 < \dots < i_p \leq N\}$, $\mathbf{j} = \{1 \leq j_1 < \dots < j_p \leq N\}$,

$$M_{\mathbf{ij}}(u) = \sum_{\tau \in S_p} (-1)^\tau L_{i_1, j_{\tau(1)}}(u) \dots L_{i_p, j_{\tau(p)}}(u - (p-1)h).$$

$$A_p(u) = M_{\mathbf{ii}}(u) \text{ for } \mathbf{i} = \{1, \dots, p\}, \quad A_p(u) = 1 + h \sum_{s=1}^{\infty} A_{p,s} u^{-s}.$$

The Gelfand-Zetlin subalgebra $\mathcal{A} \subset \mathcal{Y}$ is generated by all $A_{p,s}$.

Take distinct nonzero q_1, \dots, q_N .

$$\begin{aligned} B_p^q(u) &= \sum_{\mathbf{i}=\{1 \leq i_1 < \dots < i_p \leq N\}} q_{i_1} \dots q_{i_p} M_{\mathbf{ii}}(u) \\ &= \sigma_p(q_1, \dots, q_N) + h \sum_{s=1}^{\infty} B_{p,s}^q u^{-s}, \end{aligned}$$

where $\sigma_p(q_1, \dots, q_N)$ is the p -th elementary symmetric function.

The Bethe subalgebra $\mathcal{B}^q \subset \mathcal{Y}$ is generated by all $B_{p,s}^q$.

The subalgebras \mathcal{A} and \mathcal{B}^q are commutative.

The subalgebra \mathcal{B}^q depends only on the ratios $q_2/q_1, \dots, q_N/q_{N-1}$.

\mathcal{B}^q tends to \mathcal{A} in the limit $q_{i+1}/q_i \rightarrow 0$, $i = 1, \dots, N-1$:

$$B_p(u) = q_1 \dots q_p A_p(u) + \dots$$

For any $X \in \mathcal{A}$ or $X \in \mathcal{B}^q$, $\varpi(X) = X$. Thus for any $f, g \in \frac{1}{D}\mathcal{V}^-$,

$$\langle\!\langle Xf, g \rangle\!\rangle = \langle\!\langle f, Xg \rangle\!\rangle.$$

Both \mathcal{A} and \mathcal{B}^q act on each $\frac{1}{D}\mathcal{V}_{\lambda}^-$.

Let \mathcal{A}_{λ} and \mathcal{B}_{λ}^q be the images of \mathcal{A} and \mathcal{B}^q in $\text{End}(\frac{1}{D}\mathcal{V}_{\lambda}^-)$.

Isomorphisms $H_{\lambda} \xrightarrow{\zeta_{\lambda}} \mathbb{C}[\Gamma; h]^{S_{\lambda}} \xrightarrow{\vartheta_{\lambda}} \frac{1}{D}\mathcal{V}_{\lambda}^-$. Let $\nu_{\lambda} = \vartheta_{\lambda} \zeta_{\lambda}$.

For any $f, g \in H_{\lambda}$, $\langle f, g \rangle = \langle\!\langle \nu_{\lambda}(f), \nu_{\lambda}(g) \rangle\!\rangle$.

There is an algebra isomorphism $\mu_{\lambda}: H_{\lambda} \rightarrow \mathcal{A}_{\lambda}$ such that
for any $f, g \in H_{\lambda}$,

$$\nu_{\lambda}(fg) = \mu_{\lambda}(f)\nu_{\lambda}(g).$$

There are an algebra \mathcal{H}_{λ}^q (an explicit flat deformation of H_{λ}),
 an algebra isomorphism $\mu_{\lambda}^q : \mathcal{H}_{\lambda}^q \rightarrow \mathcal{B}_{\lambda}^q$, and
 an isomorphism $\nu_{\lambda}^q : \mathcal{H}_{\lambda}^q \rightarrow \frac{1}{D}\mathcal{V}_{\lambda}^-$, such that for any $f, g \in \mathcal{H}_{\lambda}^q$,

$$\nu_{\lambda}^q(fg) = \mu_{\lambda}^q(f)\nu_{\lambda}^q(g).$$

Actually, $\nu_{\lambda}^q(f) = \mu_{\lambda}^q(f)v_{\lambda}^-$,

similarly to $\nu_{\lambda}(f) = \mu_{\lambda}(f)\nu_{\lambda}(1) = \mu_{\lambda}(f)v_{\lambda}^-$.

The quantization map $\beta_{\lambda} : H_{\lambda} \rightarrow \mathcal{H}_{\lambda}^q$, $\beta_{\lambda} = (\nu_{\lambda}^q)^{-1}\nu_{\lambda}$.

A new multiplication \bullet on H_{λ} : $\beta_{\lambda}(f \bullet g) = \beta_{\lambda}(f)\beta_{\lambda}(g)$.

- a) $f \bullet g$ is a power series in $q_2/q_1, \dots, q_N/q_{N-1}$;
- b) $f \bullet g = fg + \dots$ as $q_{i+1}/q_i \rightarrow 0$, $i = 1, \dots, N-1$;
- c) $f \bullet 1 = f$; (since $\beta_{\lambda}(1) = 1$)
- d) $\langle f \bullet g_1, g_2 \rangle = \langle g_1, f \bullet g_2 \rangle$ — the operators $f \bullet : H_{\lambda} \rightarrow H_{\lambda}$ are symmetric with respect to the Poincare pairing on $T^*\mathcal{F}_{\lambda}$;

Isomorfisms $\nu_{\lambda} : H_{\lambda} \rightarrow \frac{1}{D} \mathcal{V}_{\lambda}^-$ and $\mu_{\lambda} : H_{\lambda} \rightarrow \mathcal{A}_{\lambda}$ are related as

$$\mu_{\lambda}(f) = \nu_{\lambda} \cdot \hat{f} \cdot (\nu_{\lambda})^{-1}, \quad f \in H_{\lambda},$$

where $\hat{f} : H_{\lambda} \rightarrow H_{\lambda}$ is the operator of multiplication by f .

The operator $F = \nu_{\lambda} \cdot (f \bullet) \cdot (\nu_{\lambda})^{-1}$ is the unique element of \mathcal{B}^q such that $F v_{\lambda}^- = \mu_{\lambda}(f) v_{\lambda}^-$.

Dynamical Hamiltonians $X_1^\infty, \dots, X_N^\infty \in \mathcal{A}$, $X_1^q, \dots, X_N^q \in \mathcal{B}^q$:

$$X_i^\infty = L_{ii}^{\{2\}} - \frac{h}{2} L_{ii}^{\{1\}} (L_{ii}^{\{1\}} - 1) - h(G_{i,1} + \dots + G_{i,i-1}),$$

$$X_i^q = X_i^\infty + h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} G_{ij} + h \sum_{j=i+1}^n \frac{q_j}{q_i - q_j} G_{ij},$$

$$G_{ij} = L_{ij}^{\{1\}} L_{ji}^{\{1\}} - L_{jj}^{\{1\}} = L_{ji}^{\{1\}} L_{ij}^{\{1\}} - L_{ii}^{\{1\}}.$$

As $q_{i+1}/q_i \rightarrow 0$, $i = 1, \dots, N-1$, $X_k^q \rightarrow X_k^\infty$, $k = 1, \dots, N$.

$$\mu_{\lambda}(\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}) = X_i^{\infty},$$

$$\nu_{\lambda} \cdot ((\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}) \bullet) \cdot (\nu_{\lambda})^{-1} =$$

$$= X_i^q + h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} \min(\lambda_i, \lambda_j) + h \sum_{j=i+1}^n \frac{q_j}{q_i - q_j} \min(\lambda_i, \lambda_j).$$

The connection $\nabla_i = \kappa q_i \frac{\partial}{\partial q_i} - X_i^q$ is flat for any $\kappa \neq 0$,

$$[\nabla_i, \nabla_j] = 0, \quad i, j = 1, \dots, N.$$

The connection $\nabla_i^\bullet = \kappa q_i \frac{\partial}{\partial q_i} - (\gamma_{i,1} + \dots + \gamma_{i,\lambda_i}) \bullet$ is gauge equivalent to ∇ :

$$\nabla_i^\bullet = (\nu_{\lambda} \Theta)^{-1} \nabla_i (\nu_{\lambda} \Theta), \quad \Theta = \prod_{1 \leq i < j \leq n} (1 - q_j/q_i)^{h \min(\lambda_i, \lambda_j)},$$

and hence is flat too for any $\kappa \neq 0$: $[\nabla_i^\bullet, \nabla_j^\bullet] = 0, \quad i, j = 1, \dots, N.$

For $[f(\Gamma; \mathbf{z}; h)] \in H_{\boldsymbol{\lambda}}$,

$$\nu_{\boldsymbol{\lambda}}([f]) = \sum_{I \in \mathcal{I}_{\boldsymbol{\lambda}}} \frac{f(\mathbf{z}_I; \mathbf{z}; h)}{R(\mathbf{z}_I)} \xi_I, \quad R(\mathbf{z}_I) = \prod_{1 \leq a < b \leq N} \prod_{\substack{i \in I_a \\ j \in I_b}} (z_i - z_j).$$

$$A_p(u) \xi_I = \xi_I \prod_{a=1}^p \prod_{i \in I_a} \left(1 + \frac{h}{u - z_i}\right).$$

$$\prod_{a=1}^p \prod_{i=1}^{\lambda_a} \left(1 + \frac{h}{u - \gamma_{ai}}\right) = 1 + h \sum_{s=1}^{\infty} f_{p,s} u^{-s}$$

The isomorphism $\mu_{\boldsymbol{\lambda}} : H_{\boldsymbol{\lambda}} \rightarrow \mathcal{A}_{\boldsymbol{\lambda}}$ is given by $\mu_{\boldsymbol{\lambda}}([f_{p,s}]) = A_{p,s}$, where

$$A_p(u) = 1 + h \sum_{s=1}^{\infty} A_{p,s} u^{-s}.$$

Let $E_{ij} \in \text{End}(\mathbb{C}^N)$ be matrix units: $E_{ij} v_k = \delta_{jk} v_i$.

Let $E_{ij}^{(a)} = 1^{\otimes(a-1)} \otimes E_{ij} \otimes 1^{\otimes(n-a)} \in \text{End}(V)$.

Let $e_{ij} = \sum_{a=1}^n E_{ij}^{(a)}$ be the generators of the \mathfrak{gl}_N -action on V .

Flip map $P = \sum_{i,j=1}^N E_{ij} \otimes E_{ji} \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$, $L_{ij}^{\{1\}} = E_{ji}$,

$$X_i^\infty = \sum_{a=1}^n z_a E_{ii}^{(a)} + \frac{h}{2} (e_{ii} - e_{ii}^2) + h \sum_{j=1}^N \sum_{1 \leq b < a \leq n} E_{ij}^{(a)} E_{ji}^{(b)} - h \sum_{j=1}^{i-1} G_{ij},$$

$$X_i^q = X_i^\infty + h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} G_{ij} + h \sum_{j=i+1}^n \frac{q_j}{q_i - q_j} G_{ij},$$

$$G_{ij} = e_{ij} e_{ji} - e_{ii} = e_{ji} e_{ij} - e_{jj}$$

The vector v_{λ}^- has the property: $e_{ij} v_{\lambda}^- = 0$ if $\lambda_i \leq \lambda_j$.

Algebra \mathcal{H}_{λ}^q .

$$\mathcal{H}_{\lambda}^q = \mathbb{C}[\Gamma; \mathbf{z}; h]^{S_{\lambda_1} \times \dots \times S_{\lambda_N} \times S_n} / \langle q\text{-relations} \rangle,$$

where $\Gamma = (\gamma_{i,j})_{i=1,\dots,N, j=1,\dots,\lambda_i}$, $\mathbf{z} = (z_1, \dots, z_n)$, and the relations are as follows. Let

$$W^q(u) = \det \left(q_i^{N-j} \prod_{k=1}^{\lambda_i} (u - \gamma_{ik} + h(i-j)) \right)_{i,j=1}^N.$$

The relations are: $W^q(u) = \prod_{1 \leq i < j \leq N} (q_i - q_j) \prod_{a=1}^n (u - z_a)$ identically

in u . In the limit $q_{i+1}/q_i \rightarrow 0$, $i = 1, \dots, N-1$, the relations in \mathcal{H}_{λ}^q turn into

$$\prod_{a=1}^N \prod_{i=1}^{\lambda_a} (u - \gamma_{ai}) = \prod_{a=1}^n (u - z_a),$$

which are the relations in H_{λ} .

The isomorphism $\mu_{\lambda}^q : \mathcal{H}_{\lambda}^q \rightarrow \mathcal{B}_{\lambda}^q$ is as follows.

$$\widehat{W}^q(u, x) = \det \left(q_i^{N-j} \prod_{k=1}^{\lambda_i} (u - \gamma_{ik} + h(i-j)) \right)_{i,j=0}^N,$$

where $q_0 = x$ and $\lambda_0 = 0$. Clearly,

$$\widehat{W}^q(u, x) = x^N W^q(u) + \dots + (-1)^N W^q(u + h).$$

Expand in u^{-1} :

$$\frac{\widehat{W}^q(u, x)}{W^q(u)} = \prod_{i=1}^N (x - q_i) + h \sum_{p=1}^N \sum_{s=1}^{\infty} (-1)^p W_{p,s}^q x^{N-p} u^{-s}.$$

The isomorphism $\mu_{\lambda}^q : \mathcal{H}_{\lambda}^q \rightarrow \mathcal{B}_{\lambda}^q$ is given by $\mu_{\lambda}^q(W_{p,s}^q) = B_{p,s}^q$, where

$$B_p^q(u) = 1 + h \sum_{s=1}^{\infty} B_{p,s}^q u^{-s}.$$