On Bogomolny decomposition and some solutions of some Skyrme-like models

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Outline

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Bogomolny equations - an introduction I

- Euler-Lagrange equations of many models in physics are nonlinear partial differential equations of second order
- but, in [Bogomolny 1976] Bogomolny derived the equations, called as Bogomolny equations - sometimes called also, as Bogomol'nyi equations (although historically, they were derived earlier in [Belavin, Polyakov, Schwarz, Tyupkin 1975], for another model - SU(2) Yang-Mills theory):

Bogomolny equations - an introduction II

1. scalar field theory - model ϕ^4 with spontaneous symmetry breaking

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2} \left(\frac{d\phi}{dx} \right)^2 + \frac{\lambda}{2} (\phi^2 - \gamma^2)^2 \right) dx, \qquad (1)$$

$$\phi(x) \in \mathbb{R}, \quad \lim_{x \to \pm\infty} \phi(x) = \pm\gamma$$

1. Euler-Lagrange equations for this model

$$\frac{d^2\phi}{dx^2} = 2\lambda\phi(\phi^2 - \gamma^2) \qquad (2)$$

Bogomolny equations - an introduction III

2. we may avoid solving of them, namely we write the formula for E in (1), as follows

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2} \left(\frac{d\phi}{dx} + \sqrt{\lambda}(\phi^2 - \gamma^2) \right)^2 - \underbrace{\sqrt{\lambda} \; \frac{d\phi}{dx}(\phi^2 - \gamma^2)}_{\text{total derivative of } \sqrt{\lambda}(\frac{\phi^3}{3} - \gamma^2 \phi)} \right) dx, \tag{3}$$

Bogomolny equations - an introduction IV

3. we integrate the underbraced term in (3)

$$E = \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{d\phi}{dx} + \sqrt{\lambda}(\phi^2 - \gamma^2)\right)^2 dx + \frac{2\sqrt{\lambda}}{3}\gamma^2 \mid Q \mid, \tag{4}$$

$$Q = \phi(\infty) - \phi(-\infty),$$

where Q - topological charge.

Bogomolny equations - an introduction V

4. now we require reaching the minimum by the functional (4), so the first term must vanish

$$\frac{d\phi}{dx} = \sqrt{\lambda}(\gamma^2 - \phi^2) \tag{5}$$

The very-known solution of (5), so called "kink"

$$\phi(x) = \gamma \tanh\left(\gamma\sqrt{\lambda}(x - x_0)\right) \tag{6}$$

So, the following inequality (Bogomolny bound) is satisfied

$$E \geq E_{min} = \frac{2\sqrt{\lambda}}{3}\gamma^2 \mid Q \mid \tag{7}$$

where E_{min} - the minimum of the functional (4).

Outline

Baby Skyrme model - an introduction I

I. Skyrme model - very interesting model, possesses solitonic solutions, useful for describing phenomena in world of baryons; good description of low-energy physics of strong interactions, [Makhankov etal. 1989].

II. baby Skyrme model - an analogical model (on plane) to the Skyrme model in three-dimensional space.

III. the target space of Skyrme model is $SU(2)$, [Skyrme 1961], [Skyrme 1962], [Skyrme 1971] $\Rightarrow$ the target space of baby Skyrme model is S^2.

IV. in these both models: Skyrme and baby Skyrme, static field configurations can be classified topologically by their winding numbers.

Baby Skyrme model - an introduction II

- analogically to the Skyrme model, the baby Skyrme model includes:
  1. the quadratic term i.e. the term of nonlinear $O(3)$ sigma model,
  2. the quartic term - an analogue of the Skyrme term and necessary in order to avoid the consequences of Derrick-Hobart theorem
     and
  3. the potential - its presence in the case of static field configurations with finite energy, in baby Skyrme model, is necessary. However, the form of this potential - not restricted.

Baby Skyrme model - an introduction III

VI the lagrangian of baby Skyrme model, [Adam etal. 2009]:

$$\mathcal{L} = \partial_\mu \vec{S} \cdot \partial^\mu \vec{S} - \beta(\partial^\mu \vec{S} \times \partial^\nu \vec{S})^2 - V(\vec{S}), \tag{8}$$

where $|\vec{S}|^2 = 1$.

VII we consider the energy functional for restricted baby Skyrme model in (2+0) dimensions (the static σ term is absent), of the following form, [Adam etal. 2010]

$$H = \frac{1}{2}\int d^2x \mathcal{H} = \frac{1}{2}\int d^2x \left(\frac{\beta}{4}(\epsilon_{ij}\partial_i \vec{S} \times \partial_j \vec{S})^2 + \gamma^2 V(\vec{S}) \right), \tag{9}$$

where we assume **nothing** about the form of the potential V (of course, $V \in \mathcal{C}$).

Outline

Our goals I

- we want to derive Bogomolny equations for ungauged baby Skyrme model in dimensions: (2+0) and (1+1) and for gauged baby Skyrme model in (2+0) dimensions
- in contrary to [Adam etal. 2010] (where only special form of potential was investigated), [Speight 2010] (where special class of potentials was investigated) and [Adam etal. 2012]: we derive Bogomolny equations (we call them as Bogomolny decomposition), by applying so called, **concept of strong necessary conditions** (firstly presented in [Sokalski 1979] and developed in [Sokalski etal. 2001], [Sokalski etal.II 2001], [Sokalski etal. 2002]), for ungauged and gauged versions of restricted baby Skyrme model.

Outline

The concept of strong necessary conditions I

- from the extremum principle, applied to the functional

$$\Phi[u] = \int_{E^2} F(u, u_{,x}, u_{,t})\ dxdt, \tag{10}$$

follow the Euler-Lagrange equations

$$F_{,u} - \frac{d}{dx} F_{,u_{,x}} - \frac{d}{dt} F_{,u_{,t}} = 0, \tag{11}$$

The concept of strong necessary conditions II

- instead of (11) we consider strong necessary conditions, [Sokalski 1979], [Sokalski etal. 2001], [Sokalski etal.II 2001], [Sokalski etal. 2002]

$$F_{,u} = 0, \tag{12}$$
$$F_{,u_{,t}} = 0, \tag{13}$$
$$F_{,u_{,x}} = 0, \tag{14}$$

where $F_{,u} \equiv \frac{\partial F}{\partial u}$, etc.

- all solutions of the system of the equations (12) - (14) satisfy the Euler-Lagrange equation (11)
BUT

The concept of strong necessary conditions III

- these solutions, if they exist, are very often trivial.
- a cure:

  A we make gauge transformation of the functional (10)

$$\Phi \to \Phi + Inv, \tag{15}$$

  where *Inv* is such functional that its local variation with respect to $u(x,t)$ vanishes: $\delta Inv \equiv 0 \Longrightarrow$ E.-L. equations are **invariant** with respect to the gauge transformation (15).

  B **non-invariance** of the strong necessary conditions (12) - (14) with respect to the gauge transformation (15) $\Longrightarrow$ some non-trivial solutions are possible

- now we apply the strong necessary conditions (12) - (14) to the gauged functional: $\tilde{\Phi} = \Phi + Inv$

The concept of strong necessary conditions IV

- we obtain so called *dual equations*.

Outline

Ungauged restricted baby Skyrme model I

- After making stereographic projection

$$\vec{S} = \left[\frac{\omega + \omega^*}{1 + \omega\omega^*}, \frac{-i(\omega - \omega^*)}{1 + \omega\omega^*}, \frac{1 - \omega\omega^*}{1 + \omega\omega^*}\right], \tag{16}$$

where $\omega = \omega(x, y) \in \mathbb{C}$ and $x, y \in \mathbb{R}$, the density of the energy functional (9) has the form

$$\mathcal{H} = -4\beta \frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})^2}{(1 + \omega\omega^*)^4} + V(\omega, \omega^*) \tag{17}$$

Ungauged restricted baby Skyrme model II

- now, we make gauge transformation, [Ł. T. S. 2012]:

$$\mathcal{H} \longrightarrow \tilde{\mathcal{H}} = -4\beta \frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})^2}{(1+\omega\omega^*)^4} + V(\omega,\omega^*) + \sum_{k=1}^{3} I_k, \qquad (18)$$

where I_k are the densities of the invariants:
$I_1 = G_1(\omega,\omega^*)(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})$ is the density of topological invariant, $I_2 = D_x G_2(\omega,\omega^*), I_3 = D_y G_3(\omega,\omega^*), D_x \equiv \frac{d}{dx}, D_y \equiv \frac{d}{dy}$
$\omega = \omega(x,y), \omega^* = \omega^*(x,y) \in \mathcal{C}^2$ and $G_k = G_k(\omega,\omega^*) \in \mathcal{C}^2$,
$(k = 1,2,3)$, are some functions, which are to be determinated.

Ungauged restricted baby Skyrme model III

- If we apply the concept of strong necessary conditions to (18), the dual equations are, as follows

$$
\begin{gathered}
\tilde{\mathcal{H}}_{,\omega} = 16\beta \frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})^2\omega^*}{(1+\omega\omega^*)^5} + V_{,\omega}(\omega,\omega^*) + \\
G_{1,\omega}(\omega,\omega^*)(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) + D_x G_{2,\omega}(\omega,\omega^*) + \\
D_y G_{3,\omega}(\omega,\omega^*) = 0,
\end{gathered} \tag{19}
$$

$$
\begin{gathered}
\tilde{\mathcal{H}}_{,\omega^*} = 16\beta \frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})^2\omega}{(1+\omega\omega^*)^5} + V_{,\omega^*}(\omega,\omega^*) + \\
G_{1,\omega^*}(\omega,\omega^*)(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) + D_x G_{2,\omega^*}(\omega,\omega^*) + \\
D_y G_{3,\omega^*}(\omega,\omega^*) = 0,
\end{gathered} \tag{20}
$$

Ungauged restricted baby Skyrme model IV

$$\tilde{\mathcal{H}}_{,\omega_{,x}} = -8\beta \frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})\omega^*_{,y}}{(1+\omega\omega^*)^4} + G_1(\omega,\omega^*)\omega^*_{,y} + G_{2,\omega} = 0, \tag{21}$$

$$\tilde{\mathcal{H}}_{,\omega_{,y}} = 8\beta \frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})\omega^*_{,x}}{(1+\omega\omega^*)^4} - G_1(\omega,\omega^*)\omega^*_{,x} + G_{3,\omega} = 0, \tag{22}$$

$$\tilde{\mathcal{H}}_{,\omega^*_{,x}} = 8\beta \frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})\omega_{,y}}{(1+\omega\omega^*)^4} - G_1(\omega,\omega^*)\omega_{,y} + G_{2,\omega^*} = 0, \tag{23}$$

Ungauged restricted baby Skyrme model V

$$\tilde{\mathcal{H}}_{,\omega^*_{,y}} = -8\beta\frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})\omega_{,x}}{(1+\omega\omega^*)^4} + G_1(\omega,\omega^*)\omega_{,x} + G_{3,\omega^*} = 0. \tag{24}$$

- Now, we need to make the equations (19) - (24) self-consistent $\Rightarrow$ the necessity of the reduction of the number of independent equations by an appropriate choice of the functions $G_k, (k = 1, 2, 3)$.
- usually, such ansatzes exist only for some special $V(\omega,\omega^*)$ $\Rightarrow$ in most cases of $V(\omega,\omega^*)$ for many nonlinear field models, the reduction of the system of corresponding dual equations, to Bogomolny equations, is impossible.

Ungauged restricted baby Skyrme model VI

- two operations (they were applied firstly in [Sokalski etal. 2002] for the cases of hyperbolic and elliptic systems of nonlinear PDE's).
  - at first, integrating the equations (19) - (20) with respect to ω and to ω^*, correspondingly. We get:

$$-4\beta\frac{(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})^2}{(1+\omega\omega^*)^4}+V(\omega,\omega^*)+G_1(\omega,\omega^*)(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})+ D_xG_2(\omega,\omega^*)+D_yG_3(\omega,\omega^*)=F(\omega_{,x},\omega_{,y},\omega^*_{,x},\omega^*_{,y}), \qquad (25)$$

where F is some function, which will be determined later.

Ungauged restricted baby Skyrme model VII

- making the equations (21) - (24) self-consistent: proper choice of the functions $G_k, k = 1, 2, 3$:
proper multiplying of the equations (21) - (24) by $\omega_{,x}, \omega_{,y}, \omega^*_{,x}, \omega^*_{,y}$, correspondingly, and adding by sides obtained equations, we get

$$-8\beta\frac{(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x})^2}{(1+\omega\omega^*)^4} + G_1(\omega,\omega^*)(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) + D_x G_2(\omega,\omega^*) = 0, \tag{26}$$

Ungauged restricted baby Skyrme model VIII

$$-8\beta\frac{(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})^2}{(1+\omega\omega^*)^4}+G_1(\omega,\omega^*)(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})+ D_yG_3(\omega,\omega^*)=0. \tag{27}$$

- Hence:

$$D_xG_2(\omega,\omega^*)=D_yG_3(\omega,\omega^*). \tag{28}$$

Ungauged restricted baby Skyrme model IX

IV now: multiplying again the equations (21) - (24) by $\omega_{,x}, \omega_{,y}, \omega^*_{,x}, \omega^*_{,y}$ and add by sides, but such, that we get

$$D_y G_2(\omega, \omega^*) = 0, \quad D_x G_3(\omega, \omega^*) = 0. \tag{29}$$

the relations (26), (27) and (29) - so called divergent representation (the divergent representation was derived firstly in [Sokalski etal. 2002] for hyperbolic system of two coupled nonlinear partial differential equations).

V Hence, and from (28)

$$G_2(\omega, \omega^*) = const, \quad G_3(\omega, \omega^*) = const. \tag{30}$$

Ungauged restricted baby Skyrme model X

VI Hence, after inserting (30) into (26), (27) and simplifying, we get:

$$\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x} = \frac{1}{8\beta}G_1(\omega,\omega^*)(1+\omega\omega^*)^4. \qquad (31)$$

VII the same result follows from (21)-(24) and all solutions of (31) satisfy the equations (21) - (24)

VIII When the equation (25) is satisfied by the solutions of (31) ? We insert (30) and (31), into the equation (25)

$$V(\omega,\omega^*) + \frac{1}{16\beta}G_1^2(\omega,\omega^*)(1+\omega\omega^*)^4 = F(\omega_{,x},\omega_{,y},\omega^*_{,x},\omega^*_{,y}). \qquad (32)$$

Ungauged restricted baby Skyrme model XI

IX Now, in order to determining function F, we compare (32) with Hamilton-Jacobi equation, [Rund 1966], [Sokalski etal. 2002]:

$$\tilde{\mathcal{H}} = 0, \tag{33}$$

where, of course $\tilde{\mathcal{H}}$ in general, for $\omega = \omega(x^\mu), \omega^* = \omega^*(x^\mu)$, $(\mu = 0, 1, 2, 3$ and $x^0 = t)$:

$$\tilde{\mathcal{H}} = \Pi_\omega \omega_{,t} + \Pi_{\omega^*} \omega^*_{,t} - \tilde{\mathcal{L}}, \tag{34}$$

where $\Pi_\omega = \tilde{\mathcal{L}}_{\omega_{,t}}, \Pi_{\omega^*} = \tilde{\mathcal{L}}_{\omega^*_{,t}}$ - canonical momenta and $\tilde{\mathcal{L}}$ - Lagrange density, gauge-transformed on the invariants $I_k, (k = 1, 2, 3)$

Ungauged restricted baby Skyrme model XII

Obviously, in our case: $\tilde{\mathcal{H}} = -\tilde{\mathcal{L}}$. Hence, by inserting into this equation, the relations (30) and (31), and taking into account (33), we get that $F = 0$. So, we get

$$V(\omega,\omega^*) = -\frac{1}{16\beta} G_1^2(\omega,\omega^*)(1+\omega\omega^*)^4. \tag{35}$$

Then, of course,

X

$$G_1 = \frac{4i\sqrt{\beta}}{(1+\omega\omega^*)^2}\sqrt{V(\omega,\omega^*)}. \tag{36}$$

Ungauged restricted baby Skyrme model XIII

XI We insert (36) in (31) and we obtain Bogomolny decomposition for the given potential $V(w, w^*)$

$$\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x} = \frac{i}{2\sqrt{\beta}}\sqrt{V(\omega,\omega^*)}(1+\omega\omega^*)^2. \qquad (37)$$

Then, the equation (37) is Bogomolny decomposition (Bogomolny equation) for restricted baby Skyrme model in (2+0) dimensions, for *arbitrary* potential.
We find an exact solution of Bogomolny decomposition (37) for $V = (\omega\omega^* - \gamma^2)^2$ - "Mexican hat" potential, i.e. it is the model with spontaneously broken symmetry.

Ungauged restricted baby Skyrme model XIV

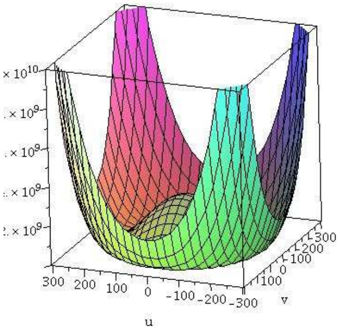

Figure : The potential

Ungauged restricted baby Skyrme model XV

We use so called "hedgehog" ansatz

$$\omega = \frac{\sin\left(f(r)\right)\cos\left(N\theta\right) + i\sin\left(f(r)\right)\sin\left(N\theta\right)}{1+\cos\left(f(r)\right)} \tag{38}$$

After inserting it into the Bogomolny decomposition, and solving obtained ODE, we get exact solution for $f(r)$ and we present here the figures of: the function $f(r)$, energy density and the components $S^i, i = 1, 2, 3$, for $\gamma = 5, N = 1$ and $\gamma = 5, N = 5$, correspondingly (of course, $\omega = u + iv, r^2 = x^2 + y^2$):

Ungauged restricted baby Skyrme model XVI

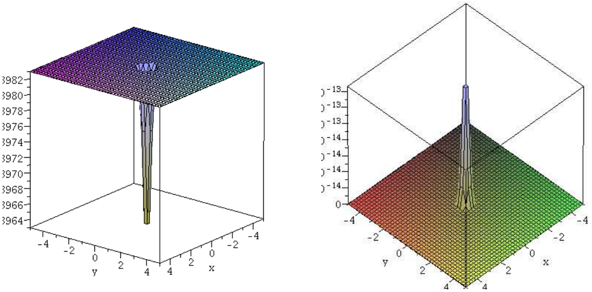

Figure : Function $f(r)$ and energy density

Ungauged restricted baby Skyrme model XVII

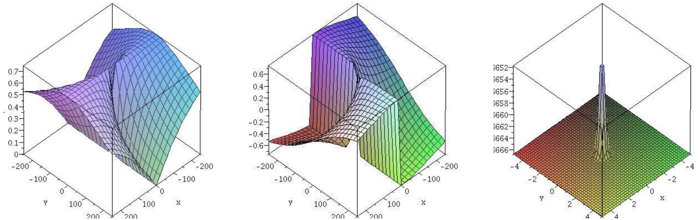

Figure : Components of vector $\vec{S}$

Ungauged restricted baby Skyrme model XVIII

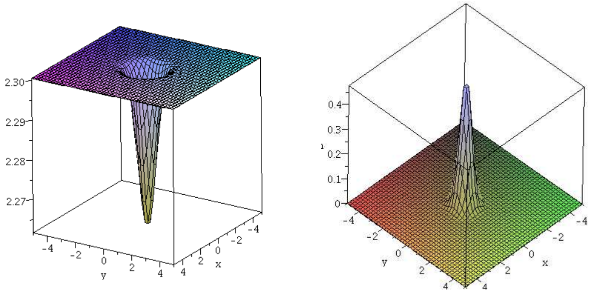

Figure : Function $f(r)$ and energy density

Ungauged restricted baby Skyrme model XIX

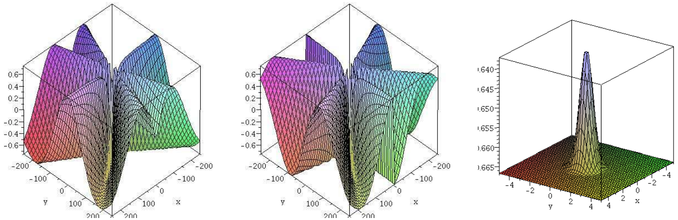

Figure : Components of vector $\vec{S}$

Gauged restricted baby Skyrme model I

- full gauged baby Skyrme model

$$\mathcal{L} = D_\mu \vec{S} \cdot D^\mu \vec{S} + \frac{\lambda^2}{4}(D^\mu \vec{S} \times D^\nu \vec{S})^2 + (1 - \vec{n} \cdot \vec{S}) + F_{\mu\nu}^2, \quad (39)$$

- gauged restricted baby Skyrme model with the potential $V = (1 - \vec{n} \cdot \vec{S})$, the special case of gauged full baby Skyrme model (8), when the $O(3)$-like term is absent, [Ł. T. S. II 2012]

$$\mathcal{L} = \frac{\lambda^2}{4}(D_\mu \vec{S} \times D_\nu \vec{S})^2 + F_{\mu\nu}^2 + (1 - \vec{n} \cdot \vec{S}), \quad (40)$$

where $\vec{S}$ is three-component vector, such that $|\ \vec{S}\ |^2 = 1$ and $D_\mu \vec{S} = \partial_\mu \vec{S} + A_\mu (\vec{n} \times \vec{S})$ is covariant derivative of vector field $\vec{S}$.

Gauged restricted baby Skyrme model II

- we consider gauged restricted baby Skyrme model in (2+0) dimensions, [Ł. T. S. II 2012]

$$H = \frac{1}{2}\int d^2x\ \mathcal{H} = \frac{1}{2}\int d^2x\left(\frac{\lambda^2}{4}(\epsilon_{ij}D_i\vec{S}\times D_j\vec{S})^2 + F_{\mu\nu}^2 + \gamma^2 V(1-\vec{n}\cdot\vec{S})\right), \tag{41}$$

where $x_1 = x,\ x_2 = y$ and $i,j = 1,2$.

- we make the stereographic projection

$$\vec{S} = \left[\frac{\omega+\omega^*}{1+\omega\omega^*}, \frac{-i(\omega-\omega^*)}{1+\omega\omega^*}, \frac{1-\omega\omega^*}{1+\omega\omega^*}\right], \tag{42}$$

where $\omega = \omega(x,y) \in \mathbb{C}$ and $x,y \in \mathbb{R}$.

Gauged restricted baby Skyrme model III

- Then (after rescalling, the constants λ_1, λ_2 have been appeared, instead of λ and γ has been included in V):

$$\mathcal{H} = \frac{1}{16\lambda_1} N_1^2 (1 + \omega\omega^*)^4 + \lambda_2 (A_{2,x} - A_{1,y})^2 + V\left(\frac{2\omega\omega^*}{1 + \omega\omega^*}\right), \tag{43}$$

where: $N_1 = \frac{8\lambda_1}{(1+\omega\omega^*)^4}[i(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) - A_1(\omega_{,y}\omega^* + \omega\omega^*_{,y}) + A_2(\omega_{,x}\omega^* + \omega\omega^*_{,x})]$.

Gauged restricted baby Skyrme model IV

- The Euler-Lagrange equations for this model are, as follows

$$\frac{d}{dx}[N_1(i\omega^*_{,y}+A_2\omega^*)]+\frac{d}{dy}[N_1(-i\omega^*_{,x}-A_1\omega^*)]+ \frac{1}{4\lambda_1}N_1^2\omega^*(1+\omega\omega^*)^3-N_1(-A_1\omega^*_{,y}+A_2\omega^*_{,x})- V'\left(\frac{2\omega\omega^*}{1+\omega\omega^*}\right)\frac{2\omega^*}{(1+\omega\omega^*)^2}=0, c.c. \tag{44}$$

$$-2\lambda_2\frac{d}{dy}(A_{2,x}-A_{1,x})+N_1\cdot(\omega_{,y}\omega^*+\omega\omega^*_{,y})=0$$

$$2\lambda_2\frac{d}{dx}(A_{2,x}-A_{1,x})-N_1\cdot(\omega_{,x}\omega^*+\omega\omega^*_{,x})=0$$

Gauged restricted baby Skyrme model V

- beside the lagrangian, in order to apply concept of strong necessary conditions, we need also topological invariant (**also gauge invariance required**), its density, [Schroers 1995], [Yang 2001]:

$$I_1 = \vec{S} \cdot D_1 \vec{S} \times D_2 \vec{S} + F_{12}(1 - \vec{n} \cdot \vec{S}), \tag{45}$$

- after making the stereographic projection (42), we have:

$$I_1 = \frac{1}{(1+\omega\omega^*)^2}[2(i(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) - A_1(\omega_{,y}\omega^* + \omega\omega^*_{,y}) + A_2(\omega_{,x}\omega^* + \omega\omega^*_{,x}))] + \frac{2\omega\omega^*}{1+\omega\omega^*}(A_{2,x} - A_{1,y}). \tag{46}$$

Gauged restricted baby Skyrme model VI

- it is useful to generalize the above expression such that there by the term $A_{2,x} - A_{1,y}$, some function of the argument $\frac{2\omega\omega^*}{1+\omega\omega^*}$ may be placed

$$I_1 = \lambda_4 \Big\{ \frac{1}{(1+\omega\omega^*)^2} [2G_1' \cdot (i(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) - A_1(\omega_{,y}\omega^* + \omega\omega^*_{,y}) + A_2(\omega_{,x}\omega^* + \omega\omega^*_{,x}))] + G_1 \cdot (A_{2,x} - A_{1,y}) \Big\}, \tag{47}$$

where $\lambda_4 = const, G_1 = G_1(\frac{2\omega\omega^*}{1+\omega\omega^*})$ and G_1' denotes the derivative of the function G_1 with respect to its argument: $\frac{2\omega\omega^*}{1+\omega\omega^*}$.

Gauged restricted baby Skyrme model VII

- we make the following gauge transformation

$$\mathcal{H} \longrightarrow \tilde{\mathcal{H}} = \frac{1}{16\lambda_1} N_1^2 (1+\omega\omega^*)^4 + \lambda_2 (A_{2,x} - A_{1,y})^2 + V\left(\frac{2\omega\omega^*}{1+\omega\omega^*}\right) + \sum_{k=1}^{3} I_k, \tag{48}$$

where $N_1 = \frac{8\lambda_1}{(1+\omega\omega^*)^4}[i(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) - A_1(\omega_{,y}\omega^* + \omega\omega^*_{,y}) + A_2(\omega_{,x}\omega^* + \omega\omega^*_{,x})]$,
I_1 is given by (47),
$I_2 = D_x G_2(\omega, \omega^*), I_3 = D_y G_3(\omega, \omega^*), D_x \equiv \frac{d}{dx}, D_y \equiv \frac{d}{dy}$ and
$G_k \in \mathcal{C}^2$, $(k = 1, 2, 3)$, are some functions, which are to be determinated.

Gauged restricted baby Skyrme model VIII

- After applying the concept of strong necessary conditions to (18), we obtain the dual equations

$$
\begin{gathered}
\omega : -\frac{1}{4\lambda_1} N_1^2 \omega^* (1+\omega\omega^*)^3 + N_1(-A_1\omega^*_{,y} + A_2\omega^*_{,x}) + V' \frac{2\omega^*}{(1+\omega\omega^*)^2} + \\
\lambda_4 \Big\{ G_1'' \frac{N_1\omega^*}{2\lambda_1} + \frac{2G_1'(-A_1\omega^*_{,y} + A_2\omega^*_{,x})}{(1+\omega\omega^*)^2} - \\
G_1' \frac{N_1}{2\lambda_1}(1+\omega\omega^*)\omega^* + G_1' \frac{2\omega^*}{(1+\omega\omega^*)^2}(A_{2,x} - A_{1,y}) \Big\} + \\
D_x G_{2,\omega} + D_y G_{3,\omega} = 0
\end{gathered} \tag{49}
$$

Gauged restricted baby Skyrme model IX

$$\omega^{*}: -\frac{1}{4\lambda_1}N_1^2\omega(1+\omega\omega^{*})^3 + N_1(-A_1\omega_{,y} + A_2\omega_{x}) + V'\frac{2\omega}{(1+\omega\omega^{*})^2} +$$

$$\lambda_4\Bigg\{G_1''\frac{N_1\omega}{2\lambda_1} + \frac{2G_1'(-A_1\omega^{*}_{,y} + A_2\omega^{*}_{,x})}{(1+\omega\omega^{*})^2} -$$

$$G_1'\frac{N_1}{2\lambda_1}(1+\omega\omega^{*})\omega + G_1'\frac{2\omega}{(1+\omega\omega^{*})^2}(A_{2,x} - A_{1,y})\Bigg\} +$$

$$D_x G_{2,\omega^{*}} + D_y G_{3,\omega^{*}} = 0 \tag{50}$$

Gauged restricted baby Skyrme model X

$$\omega_{,x} : N_1(i\omega^*_{,y} + A_2\omega^*) + \frac{2\lambda_4 G_1'(i\omega^*_{,y} + A_2\omega^*)}{(1+\omega\omega^*)^2} + G_{2,\omega} = 0, \quad (51)$$

$$\omega_{,y} : N_1(-i\omega^*_{,x} - A_1\omega^*) + \frac{2\lambda_4 G_1'(-i\omega^*_{,x} - A_1\omega^*)}{(1+\omega\omega^*)^2} + G_{3,\omega} = 0, \quad (52)$$

$$\omega^*_{,x} : N_1(-i\omega_{,y} + A_2\omega) + \frac{2\lambda_4 G_1'(-i\omega_{,y} + A_2\omega)}{(1+\omega\omega^*)^2} + G_{2,\omega^*} = 0, \quad (53)$$

Gauged restricted baby Skyrme model XI

$$\omega^*_{,y}: N_1(i\omega_{,x} - A_1\omega) + \frac{2\lambda_4 G_1'(i\omega_{,x} - A_1\omega)}{(1+\omega\omega^*)^2} + G_{3,\omega^*} = 0, \tag{54}$$

$$A_1: N_1(-\omega_{,y}\omega^* - \omega\omega^*_{,y}) + \frac{2\lambda_4 G_1'(-\omega_{,y}\omega^* - \omega\omega^*_{,y})}{(1+\omega\omega^*)^2} = 0, \tag{55}$$

$$A_2: N_1(\omega_{,x}\omega^* + \omega\omega^*_{,x}) + \frac{2\lambda_4 G_1'(\omega_{,x}\omega^* + \omega\omega^*_{,x})}{(1+\omega\omega^*)^2} = 0, \tag{56}$$

$$A_{1,y}: -2\lambda_2(A_{2,x} - A_{1,y}) - \lambda_4 G_1 = 0, \tag{57}$$

Gauged restricted baby Skyrme model XII

$$A_{2,x} : 2\lambda_2(A_{2,x} - A_{1,y}) + \lambda_4 G_1 = 0, \tag{58}$$

where $N_1 = \frac{8\lambda_1}{(1+\omega\omega^*)^4}[i(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) - A_1(\omega_{,y}\omega^* + \omega\omega^*_{,y}) + A_2(\omega_{,x}\omega^* + \omega\omega^*_{,x})]$ and V', G_1', G_1'' denote the derivatives of the functions V and G_1 with respect to their argument: $\frac{2\omega\omega^*}{1+\omega\omega^*}$.

- making the equations (49) - (58) self-consistent:

Gauged restricted baby Skyrme model XIII

- we put:

$$G_1' = -\frac{N_1}{2\lambda_4}(1+\omega\omega^*)^2, \tag{59}$$

$$A_{2,x} - A_{1,y} = -\frac{\lambda_4}{2\lambda_2} G_1 \left(\frac{2\omega\omega^*}{1+\omega\omega^*}\right), \tag{60}$$

$$G_2 = const, \quad G_3 = const, \tag{61}$$

where $N_1 = \frac{8\lambda_1}{(1+\omega\omega^*)^4}[i(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) - A_1(\omega_{,y}\omega^* + \omega\omega^*_{,y}) + A_2(\omega_{,x}\omega^* + \omega\omega^*_{,x})]$.

Gauged restricted baby Skyrme model XIV

- then, the equations (51)-(58) become the tautologies and we have the candidate for Bogomolny decomposition:

$$\frac{4\lambda_1[i(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}) - A_1(\omega_{,y}\omega^* + \omega\omega^*_{,y}) + A_2(\omega_{,x}\omega^* + \omega\omega^*_{,x})]}{\lambda_4(1+\omega\omega^*)^2} = -G'_1,$$

$$2\lambda_2(A_{2,x} - A_{1,y}) + \lambda_4 G_1\left(\frac{2\omega\omega^*}{1+\omega\omega^*}\right) = 0. \tag{62}$$

Gauged restricted baby Skyrme model XV

- III When the equations (49)-(50) are satisfied, if (62) hold ? We insert (59)-(61) into (49)-(50). We get the system of ordinary differential equations for V and the solution of it is:

$$V = \frac{\lambda_4^2}{4}\left(\frac{1}{\lambda_1}(G_1')^2 + \frac{1}{\lambda_2}G_1^2\right). \tag{63}$$

Gauged restricted baby Skyrme model XVI

IV So, we obtain Bogomolny decomposition for gauged restricted baby Skyrme model in (2+0) dimensions

$$\frac{4\lambda_1[i(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x})-A_1(\omega_{,y}\omega^*+\omega\omega^*_{,y})+A_2(\omega_{,x}\omega^*+\omega\omega^*_{,x})]}{\lambda_4(1+\omega\omega^*)^2} = -G_1',$$

$$A_{2,x}-A_{1,y} = -\frac{\lambda_4}{2\lambda_2}G_1\left(\frac{2\omega\omega^*}{1+\omega\omega^*}\right). \tag{64}$$

for the potential $V(\frac{2\omega\omega^*}{1+\omega\omega^*})$, satisfying

$$V = \frac{\lambda_4^2}{4}\left(\frac{1}{\lambda_1}(G_1')^2+\frac{1}{\lambda_2}G_1^2\right), \tag{65}$$

Gauged restricted baby Skyrme model XVII

where $G_1 = G_1\left(\frac{2\omega\omega^*}{1+\omega\omega^*}\right) \in \mathcal{C}^2$.

Outline

The case of (1+1)-dimensions I

- the lagrangian:

$$\mathcal{L} = -4\beta\frac{(\omega_{,t}\omega^*_{,x} - \omega_{,x}\omega^*_{,t})^2}{(1+\omega\omega^*)^4} + V(\omega,\omega^*) \qquad (66)$$

- the gauge transformation of Lagrangian, [Ł. T. S. 2012]:

$$\mathcal{L} \longrightarrow \tilde{\mathcal{L}} = -4\beta\frac{(\omega_{,t}\omega^*_{,x} - \omega_{,x}\omega^*_{,t})^2}{(1+\omega\omega^*)^4} + V(\omega,\omega^*) + \sum_{k=1}^{3} I_k, \qquad (67)$$

where now: $I_1 = G_1(\omega,\omega^*)(\omega_{,t}\omega^*_{,x} - \omega_{,x}\omega^*_{,t}), I_2 = D_t G_2(\omega,\omega^*), I_3 = D_x G_3(\omega,\omega^*), D_t \equiv \frac{d}{dt}, D_x \equiv \frac{d}{dx},$

The case of (1+1)-dimensions II

$\omega = \omega(t,x), \omega^* = \omega^*(t,x) \in \mathcal{C}^2$ and $G_k = G_k(\omega,\omega^*) \in \mathcal{C}^2$, $(k = 1,2,3)$, are some functions, which are to be determinated.

- Further computations are analogical to the computations in the case of (2+0)-dimensions, i.e. dual equations have very similar form and $G_2 = const, G_3 = const$ and:

$$\omega_{,t}\omega^*_{,x} - \omega_{,x}\omega^*_{,t} = \frac{1}{8\beta}G_1(\omega,\omega^*)(1+\omega\omega^*)^4. \qquad (68)$$

but with one difference: the Hamilton-Jacobi equation has now another form. Namely, let us remind [Rund 1966], [Sokalski etal. 2002]:

$$\tilde{\mathcal{H}} = 0, \qquad (69)$$

The case of (1+1)-dimensions III

where, of course $\tilde{\mathcal{H}}$ in general, for $\omega = \omega(x^\mu), \omega^* = \omega^*(x^\mu)$, ($\mu = 0, 1, 2, 3$ and $x^0 = t$):

$$\tilde{\mathcal{H}} = \Pi_\omega \omega_{,t} + \Pi_{\omega^*} \omega^*_{,t} - \tilde{\mathcal{L}}, \tag{70}$$

$$\Pi_\omega = \tilde{\mathcal{L}}_{,\omega_{,t}}, \Pi_{\omega^*} = \tilde{\mathcal{L}}_{,\omega^*_{,t}}. \tag{71}$$

Obviously, in the current case:

$$\tilde{\mathcal{H}} = -4\beta \frac{(\omega_{,t}\omega^*_{,x} - \omega_{,x}\omega^*_{,t})^2}{(1+\omega\omega^*)^4} - V(\omega, \omega^*) = 0. \tag{72}$$

The case of (1+1)-dimensions IV

- After taking into account (74) and (72), we have

$$V(\omega,\omega^*)+\frac{1}{16\beta}G_1^2(\omega,\omega^*)(1+\omega\omega^*)^4=0. \tag{73}$$

Thus, we have obtained the same relation between the potential, as in the case of (2+0)-dimensions. The Bogomolny decomposition for this case has the form:

$$\omega_{,t}\omega^*_{,x}-\omega_{,x}\omega^*_{,t}=\frac{i}{2\sqrt{\beta}}\sqrt{V(\omega,\omega^*)}(1+\omega\omega^*)^2. \tag{74}$$

Summary I

- The Bogomolny decomposition (the system of Bogomolny equations) has been derived, by using the concept of strong necessary conditions in:
  (2+0)-dimensions, for:
  - ungauged restricted baby Skyrme model, for arbitrary form of the potential, in contrary to [Adam etal. 2010] (where only special form of potential was investigated), [Speight 2010] (where special class of potentials was investigated),
  - gauged restricted baby Skyrme model, the “gauging” of the model causes the condition for the potential in contrary to ungauged model

Summary II

(1+1)-dimensions for ungauged restricted baby Skyrme model, for arbitrary form of the potential.

- The figures of example exact solution of Bogomolny decomposition and corresponding energy densities, for ungauged restricted baby Skyrme model in (2+0)-dimensions, have been presented.
- further investigation of other Skyrme-like models and the further solutions of found Bogomolny decompositions and their physical features: work in progress.

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