Dunkl operators as covariant derivatives in a quantum principal bundle

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Coxeter groups - a bit of Algebra

We consider $\mathbb{R}^N$ with its standard inner product:

$$\langle x, y\rangle = \sum_{j=1}^{N} x_j y_j$$

For $0 \neq \alpha \in \mathbb{R}^N$, let σ_α be the orthogonal reflection in the hyperplane H_α orthogonal to α. So, with the action on the right, we have:

- $x\sigma_\alpha = x$ for all $x \in H_\alpha$,
- $\alpha\sigma_\alpha = -\alpha$.

$\sigma_\alpha \in O(\mathbb{R}^N) =$ Orthogonal Group.

Definition 0.1 *A finite set $\mathcal{R}$ of non-zero vectors in $\mathbb{R}^N$ is called a* **root system** *if for all $\alpha \in \mathcal{R}$*

1. $\mathbb{R}\alpha \cap \mathcal{R} = \{\alpha, -\alpha\}$,
2. $(\mathcal{R})\sigma_\alpha = \mathcal{R}$.

$G \equiv$ subgroup of $O(\mathbb{R}^N)$ generated by σ_α for $\alpha \in \mathcal{R}$.

G is finite and so we call it a ***finite Coxeter group***.

There exist non-trivial examples!
(Symmetric groups, dihedral groups, etc.)

Dunkl operators - Analysis

Any G-invariant function $\kappa : \mathcal{R} \to [0, \infty)$ is called a *multiplicity function.*

Take $j \in \{1, 2, \ldots, N\}$ and κ as above. Define

$$T_{j,\kappa} f(x) := \frac{\partial f}{\partial x_j} + \frac{1}{2} \sum_{\alpha \in \mathcal{R}} \frac{\kappa(\alpha)}{\langle \alpha, x \rangle} \big(f(x) - f(x\sigma_\alpha) \big) \alpha_j$$

to be the ***Dunkl operator*** for the j-th coordinate. Here $\alpha = (\alpha_1, \ldots, \alpha_j, \ldots, \alpha_N)$.

These are not local operators for non-zero κ.

One says Dunkl operators are deformations of directional and partial derivatives, not perturbations. They break the symmetry group from the orthogonal group $O(\mathbb{R}^N)$ ($\kappa \equiv 0$) down to the Coexter group G (non-zero κ).

Theorem 0.1 *The family of operators*

$$\{T_{j,\kappa} \mid j = 1, 2, \ldots, N\}$$

is commutative.

This theorem is a bit of a surprise. It was proved by Dunkl in his original paper on this subject (1989).

Definition 0.2 *The* ***Dunkl Laplacian*** *is* $\Delta_\kappa := \sum_{j=1}^N T_{j,\kappa}^2$. *Again: a non-local operator.*

Dunkl miscelanea - Analysis

What we can do with this theory:

1. Generalized exponential function or **Dunkl kernel**. (Simultaneous eigenfunction of all the $T_{j,\kappa}$.)
2. Dunkl version of the Fourier transform (also known as the **Dunkl transform**).
3. Dunkl heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta_\kappa u$ and its associated **Dunkl heat kernel**.
4. Applications to Calogero-Moser models.
5. Dunkl generalization of Brownian motion.
6. Self-adjoint realizations of Dunkl position and Dunkl momentum operators in $L^2(\mathbb{R}^N, m_\kappa)$, where m_κ is a measure on $\mathbb{R}^N$ absolutely continuous with respect to Lebesgue measure. (Another quantization!)
7. Self-adjoint realizations of Dunkl position and Dunkl momentum operators in a Hilbert space of holomorphic functions $\mathbb{C}^N \to \mathbb{C}$. (Another quantization?) (A generalized Segal-Bargmann space)
8. A canonical unitary isomorphism a la Segal-Bargmann from $L^2(\mathbb{R}^N, m_\kappa)$ onto the above Hilbert space of holomorphic functions. (The same quantization!)

Quantum Principal Bundles
(Non-commutative Geometry)

All algebras are over the field of complex numbers $\mathbb{C}$, are associative and have an identity element 1.

A **quantum principal bundle** (QPB) $P = (\mathcal{B}, \mathcal{A}, F)$ consists of the following objects.

- A $*$-algebra $\mathcal{B}$. ('Functions' on the total space.)
- A quantum group $\mathcal{A}$ which is a $*$-algebra. (Compact Matrix Pseudogroups, Woronowicz, 1987.) ('Functions' on the model fiber space.)
- A right co-action of the quantum group $\mathcal{A}$ on the total space $\mathcal{B}$, i.e., $F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$. F is also a unital $*$-homomorphism. (Unital means that $F(1) = 1 \otimes 1$ and $*$-homomorphism means that $F(b^*) = F(b)^*$.)

Furthermore, we require that these objects satisfy one more property, namely that the map $\beta : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ defined for $b_1, b_2 \in \mathcal{B}$ by

$$\beta(b_1 \otimes b_2) := b_1 F(b_2)$$

is surjective. (The right co-action F is free.)

We define $\mathcal{V} := \{b \in \mathcal{B} \mid F(b) = b \otimes 1\}$.
(The right invariant 'functions' on the total space, i.e., the 'functions' on the base space.)

Exterior Algebras on a Quantum Principal Bundle (Non-commutative Geometry)

An **($\mathbb{N}$-graded) differential calculus** on a quantum principal bundle $P = (\mathcal{B}, \mathcal{A}, F)$ is given by these objects:

1. A graded differential $*$-algebra $(\Omega(P), d_P)$ over $\mathcal{B}$. (So $d_P : \Omega^0(P) = \mathcal{B} \to \Omega^1(P)$ is a **first order differential calculus (fodc)** and $d_P : \Omega(P) \to \Omega(P)$ has degree 1.)

2. A bicovariant and $*$-covariant fodc (Γ, d) over $\mathcal{A}$.

3. An extension of the right co-action $F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ of the QPB to a right co-action $\hat{F} : \Omega(P) \to \Omega(P) \hat{\otimes} \hat{\Gamma}$ of $\Omega(P)$ over $\hat{\Gamma}$, where $\hat{\Gamma}$ is the enveloping differential calculus of the fodc (Γ, d) and $\hat{F}$ is a differential, unital $*$-homomorphism.

Moreover, we have these properties:

1. $\Omega(P)$ is generated as a graded differential calculus by $\Omega^0(P) = \mathcal{B}$, the elements of degree zero. The $\hat{F}$-invariant elements of $\Omega(P)$ is a differential $*$-subalgebra denoted by $\Omega(M)$. This is the differential calculus for the base space. We do not necessarily have that $\Omega(M)$ is generated by $\mathcal{V}$.

2. Other technical details. (See the paper on arXiv.)

Definition 0.3 *The* **horizontal forms** *are*

$$\mathfrak{hor}(P) := \{\omega \in \Omega(P) \,|\, \hat{F}(\omega) \in \Omega(P) \otimes \mathcal{A}\}.$$

Connections and Covariant Derivatives on a Quantum Principal Bundle (Non-commutative Geometry)

Consider a QPB with a given differential calculus.

Definition 0.4 *A* **connection** *is a linear map* $\omega : \Gamma_{inv} \to \Omega^1(P)$ *such that for all* $\theta \in \Gamma_{inv}$ *we have:*

- $\omega(\theta^*) = \omega(\theta)^*$ *(reality condition)*
- $(\hat{F}\omega)(\theta) = (\omega \otimes id)\mathrm{ad}(\theta) + 1 \otimes \theta$

where $\mathrm{ad} : \Gamma_{inv} \to \Gamma_{inv} \otimes \mathcal{A}$ *is the adjoint co-action of the quantum group on* Γ_{inv}*.*

The **covariant derivative** D_ω *of a connection* ω *is the linear map* $D_\omega : \mathfrak{hor}(P) \to \mathfrak{hor}(P)$ *defined for all* $\varphi \in \mathfrak{hor}(P)$ *by*

$$D_\omega(\varphi) := d_P(\varphi) - (-1)^{|\varphi|}\varphi^{(0)}\omega\pi(\varphi^{(1)}).$$

where $\hat{F}(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)}$ *is Sweedler's notation for a co-action. (*φ *a homogeneous element of degree* $|\varphi|$*.)*

The **curvature** $r_\omega : \mathcal{A} \to \mathfrak{hor}(P)$ *of the connection of* ω *is defined by*

$$r_\omega(a) := d_P\,\omega\,\pi(a) + \omega\pi a^{(1)}\,\omega\pi a^{(2)}$$

where $\phi(a) = a^{(1)} \otimes a^{(2)}$ is Sweedler's notation for the co-multiplication in the quantum group $\mathcal{A}$. Here we have also used the following definition:

Definition 0.5 *The* **quantum germ** *mapping* $\pi : \mathcal{A} \to \Gamma_{inv}$ *is defined for all* $a \in \mathcal{A}$ *by*

$$\pi(a) := \kappa(a^{(1)})d(a^{(2)}).$$

Here $\kappa : \mathcal{A} \to \mathcal{A}$ is the antipode of the quantum group. (Not to be confused with the multiplicity function.)

A specific QPB

A specific example of a QPB with a specific connection will now give us the Dunkl operators. First: the QPB.

The spaces are classical, but the differential calculi (DC) are not classical for two of the spaces:

- Total space: $P = \mathbb{R}^N \setminus (\cup_{\alpha \in \mathcal{R}} H_\alpha)$. Quantum DC.

  The standard differential calculus on P is denoted $\mathfrak{hor}(P)$ and will indeed turn out to be the horizontal forms on the total space P.

  As a graded $*$-algebra $\Omega(P) = \mathfrak{hor}(P) \otimes \Gamma_{inv}^{\wedge}$. The product, the $*$-operation and the differential on $\Omega(P)$ are defined in the paper; $\Gamma_{inv}^{\wedge}$ is defined as the quotient of the tensor algebra over Γ_{inv} by dividing out the quadratic relations $\sum_{g_1 g_2 = h} [g_1] \otimes [g_2]$ where $g_1, g_2 \in S$ and $e \neq h \notin S$. Also, $[g] = \pi(g)$ with $g \in S$ is the canonical basis of Γ_{inv}.

- Group: Coxeter group G associated with root system $\mathcal{R}$ (which acts freely on P). Quantum DC $\Gamma^{\wedge}$.
- Base space: $M = P/G \cong$ any connected component of P with DC taken to be the classical DC on M.

The DC on the quantum group $\mathcal{A}$ is completely quantum. The classical calculus of the zero dimensional compact Hausdorff differential manifold G is worthless! Following Gelfand theory, we take the quantum group to be

$$\mathcal{A} := \{f : G \to \mathbb{C}\},$$

the finite dimensional vector space of all complex valued (continuous) functions on G.

The Quantum DC for $\mathcal{A}$

$\mathcal{A}$ is commutative. All one-sided ideals are two-sided and given uniquely by $\mathcal{I} = \{f : G \to \mathbb{C} \mid f|_S = 0\}$ for some unique $S \subset G$.

By the theory of fodc, we must consider all the ideals $\mathcal{I} \subset \ker \epsilon$ where $\epsilon : \mathcal{A} \to \mathbb{C}$ is the co-unit given by

$$\epsilon(f) := f(e)$$

where $e \in G$ is the identity element. So, $\mathcal{I} \subset \ker \epsilon$ if and only if $e \in S$.

To get a $*$-invariant fodc is equivalent to $\kappa(\mathcal{I})^* = \mathcal{I}$ which in turn is equivalent to $S^{-1} = S$.

To get a bicovariant fodc is equivalent $\mathrm{ad}(\mathcal{I}) \subset \mathcal{I} \otimes \mathcal{A}$ which in turn is equivalent to $g^{-1}Sg = S$ for all $g \in G$.

For our case of a Coxeter group G we choose

$$S = \{e\} \cup \bigcup_{\alpha \in \mathcal{R}} \{\sigma_\alpha\}.$$

Since $\sigma_\alpha^{-1} = \sigma_\alpha$, we have $S^{-1} = S$. Also $g^{-1}\sigma_\alpha g = \sigma_{\alpha g}$ for all $g \in G$, which implies $g^{-1}Sg = S$.

Then $\Gamma_{inv} \cong \ker(\epsilon)/\mathcal{I}$ has canonical vector space basis given by $\{\pi(g) = [g] \mid g \in S\}$, which is also an $\mathcal{A}$-module basis for Γ.

The quantum connection

We continue considering the QPB introduced above. Let ω_f be the canonical flat connection defined for all $\theta \in \Gamma_{inv}$ by

$$\omega_f(\theta) := 1 \otimes \theta.$$

Theorem 0.2 .

Let $\alpha \in \mathcal{R}$ be a root. Define $\lambda : \Gamma_{inv} \to \mathfrak{hor}(P)$ for $x \in P$ by

$\lambda[\sigma_\alpha](x) = ih_\alpha(x)\,\alpha \quad$ (α considered as a 1 form on P)

Here $h_\alpha : P \to \mathbb{R}$ satisfies $h_{\alpha g}(x) = h_\alpha(xg^{-1})$ for $g \in G$. Then the covariant derivative D_ω associated with the connection $\omega := \omega_f + \lambda$ (called a **Dunkl connection***) for all $\phi \in \mathfrak{hor}(P)$ is*

$$D_\omega\phi(x) = D\phi(x) + \frac{i}{2}\sum_{\alpha\in\mathcal{R}} h_\alpha(x)\,(\phi(x) - \phi(x\sigma_\alpha))\,\alpha$$

where $D : \mathfrak{hor}(P) \to \mathfrak{hor}(P)$ is the standard de Rham derivative of classical differential geometry.

Corollary 0.1 *Now take $h_\alpha(x) = \kappa_\alpha / \langle \alpha, x\rangle$. Then the curvature $r_\omega \equiv 0$ and the covariant derivative is*

$$D_\omega\phi(x) = D\phi(x) + \frac{i}{2}\sum_{\alpha\in\mathcal{R}} \frac{\kappa_\alpha}{\langle \alpha, x\rangle}\,(\phi(x) - \phi(x\sigma_\alpha))\,\alpha$$

Concluding Remarks - and Another Theorem

MORAL: Without changing any part of the theories of Dunkl operators nor of QPB's we have found that Dunkl operators are a special example in the theory of QPB's. It is important to note that Dunkl operators can not be viewed as covariant derivatives in classical differential geometry, since the latter operators are local. Seen this way, Dunkl operators are a quantum phenomenon.

We have found a formula for $D_\omega \phi(x)$ for *all* elements $\phi \in \mathfrak{hor}(P)$ of any degree. In analysis Dunkl operators are usually (maybe always?) considered as acting only on elements $\phi \in \mathfrak{hor}^{(0)}(P)$, that is on smooth functions $\phi : P \to \mathbb{R}$.

Theorem 0.3 *The condition $r_\omega \equiv 0$ implies that the family of operators $\{T_{j,\kappa} | j{=}1, \ldots, N\}$ is commutative when applied to smooth functions $\phi : P \to \mathbb{R}$.*

This last result has been known since Dunkl's very first paper on these operators. However, we now have given a geometrical explanation (zero curvature) for why this turns out to be true.

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