Form factor approach to the correlation functions of integrable models

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Białowieża

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N. Kitanine, K.K. Kozlowski, J.M. Maillet, N.S., V. Terras

[1] *Form factor approach to the asymptotic behavior of correlation functions in critical models*, J. Stat. Mech. (2011) P12010

[2] *Form factor approach to dynamical correlation functions in critical models,* arXiv:1206.2630

Problem

Suppose that we have quantum Hamiltonian H

$$H|\psi\rangle = E|\psi\rangle$$

and we want to calculate two-point functions

$$G^{\mathcal{O}}(x,t) = \frac{\langle\psi|\mathcal{O}(x,t)\mathcal{O}^\dagger(0,0)|\psi\rangle}{\langle\psi|\psi\rangle}$$

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$$I = \sum_{|\psi'\rangle} \frac{|\psi'\rangle\langle\psi'|}{\|\psi'\|^2}$$

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Quantum Nonlinear Schrödinger equation

$$[\phi(x,t), \phi^\dagger(y,t)] = \delta(x-y)$$

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$$H = \int_0^L \left(\partial_x\phi^\dagger\partial_x\phi + c\phi^\dagger\phi^\dagger\phi\phi - h\phi^\dagger\phi\right)\,dx$$

$c > 0$ is a coupling constant

$h > 0$ is a chemical potential

$$\phi(x,t) = e^{iHt}\phi(x,0)e^{-iHt}, \qquad \phi^\dagger(x,t) = e^{iHt}\phi^\dagger(x,0)e^{-iHt}$$

Eigenfunctions

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$$|\psi\rangle = \int_0^L \chi_N(x_1, \ldots, x_N) \prod_{k=1}^{N} \phi^\dagger(x_k)\, dx_k |0\rangle, \qquad N = 0, 1, \ldots$$

1-D Bose gas

$$\left(-\sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} + 2c \sum_{j>k}^{N} \delta(x_j - x_k) - Nh \right) \chi_N = E\chi_N$$

Eigenfunctions

$$\chi_N(x_1, \dots, x_N) = \sum_P (-1)^{[P]} \prod_{j>k}^{N} \Big(\lambda_{P(j)} - \lambda_{P(k)} - ic\,\mathsf{sgn}(x_j - x_k)\Big) \prod_{m=1}^{N} e^{ix_m \lambda_{P(m)}}$$

The system of Bethe equations

$$e^{iL\lambda_j} = -\prod_{k=1}^{N} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}, \qquad j = 1, \dots, N$$

$$\lambda_j \neq \lambda_k, \qquad j, k = 1, \dots, N$$

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Eigenvalues

$$E = \sum_{j=1}^{N} (\lambda_j^2 - h), \qquad P = \sum_{j=1}^{N} \lambda_j$$

Form factors and correlation functions

$$\mathcal{F}^{\phi}_{\psi,\psi'} = \frac{\langle\psi|\phi(0,0)|\psi'\rangle}{\|\psi'\|\;\|\psi\|}, \qquad \mathcal{F}^{\phi^\dagger}_{\psi',\psi} = \frac{\langle\psi'|\phi^\dagger(0,0)|\psi\rangle}{\|\psi'\|\;\|\psi\|}$$

$$\|\psi\|^2 = N! \int_0^L |\chi_N(\{\lambda\}|z_1,\ldots,z_N)|^2 \prod_{m=1}^{N} dz_m$$

$$\|\psi'\|^2 = (N+1)! \int_0^L |\chi_{N+1}(\{\mu\}|z_1,\ldots,z_{N+1})|^2 \prod_{m=1}^{N+1} dz_m$$

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$$\langle\psi|\phi(0,0)|\psi'\rangle = N!\int_0^L \chi^*_N(\{\lambda\}|z_1,\ldots,z_N)\chi_{N+1}(\{\mu\}|0,z_1,\ldots,z_N)\prod_{m=1}^N dz_m$$

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$$\frac{\langle\psi|\phi(x,t)\phi^\dagger(0,0)|\psi\rangle}{\langle\psi|\psi\rangle} = \sum_{\{\mu\}} |\mathcal{F}^{\phi}_{\psi,\psi'}|^2 e^{ix\mathcal{P}-it\mathcal{E}},$$

$$\mathcal{E} = E_\psi - E_{\psi'}$$

$$\mathcal{P} = P_\psi - P_{\psi'}$$

We consider ground state correlation functions in the thermodynamic limit. This means that:

- $|\psi\rangle$ corresponds to the minimal energy;
- $L \to \infty$ at fixed density of the gas $D = N/L$.

$$G^{\phi}(x,t) \equiv \frac{\langle\psi|\phi(x,t)\phi^{\dagger}(0,0)|\psi\rangle}{\langle\psi|\psi\rangle} = \sum_{\{\mu\}} |\mathcal{F}^{\phi}_{\psi,\psi'}|^2 e^{ix\mathcal{P}-it\mathcal{E}}$$

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There exists a way to sum up form factor series directly with respect to solutions of Bethe equations $\{\mu\}$. This way leads to various multiple integral representations for correlation functions.

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Another way is to describe the excited states in terms of particles and holes. Then in the thermodynamic limit the sum over the excited states can be replaced by integration over rapidities of particles and holes μ_p and μ_h.

$$\sum_{\{\mu\}} |\mathcal{F}^{\phi}_{\psi,\psi'}|^2 e^{ix\mathcal{P}-it\mathcal{E}} \;\to\; \int |\mathcal{F}^{\phi}_{\psi,\psi'}|^2 e^{ix\mathcal{P}-it\mathcal{E}}\, d\mu_p\, d\mu_h, \qquad L\to\infty$$

$$|\mathcal{F}^{\phi}_{\psi,\psi'}|^2 \to L^{-\theta} C(\{\mu_p\};\{\mu_h\}), \qquad L\to\infty$$

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This situation is typical for critical (gapless) models. Such a senseless result arises, because form factors in critical models have no uniform thermodynamic limit.

In order to solve the problem we split the excited states into special classes $\mathbf{P}$

$$\sum_{|\psi'\rangle} |\mathcal{F}^{\phi}_{\psi,\psi'}|^2 e^{ix\mathcal{P}-it\mathcal{E}} = \sum_{\mathbf{P}} \sum_{|\psi'\rangle\in\mathbf{P}} |\mathcal{F}^{\phi}_{\psi,\psi'}|^2 e^{ix\mathcal{P}-it\mathcal{E}}$$

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The sum of form factors within one class of the excited states can be computed explicitly. It gives dressed form factor

$$|\mathcal{F}^{\phi}_{\mathbf{P}}|^2 e^{ix\mathcal{P}-it\mathcal{E}} = \sum_{|\psi'\rangle\in\mathbf{P}} |\mathcal{F}^{\phi}_{\psi,\psi'}|^2 e^{ix\mathcal{P}-it\mathcal{E}}$$

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The remaining sum over classes can be replaced dy the integration over rapidities of particles and holes.

Impenetrable bosons ($c \to \infty$)

Eigenfunctions

$$\chi_N(x_1, \dots, x_N) = \sum_P (-1)^{[P]} \prod_{j>k}^{N} \Big(\lambda_{P(j)} - \lambda_{P(k)} - ic\, \mathrm{sgn}(x_j - x_k) \Big) \prod_{m=1}^{N} e^{i x_m \lambda_{P(m)}}$$

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$$e^{iL\lambda_j} = - \prod_{k=1}^{N} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}, \qquad j = 1, \dots, N$$

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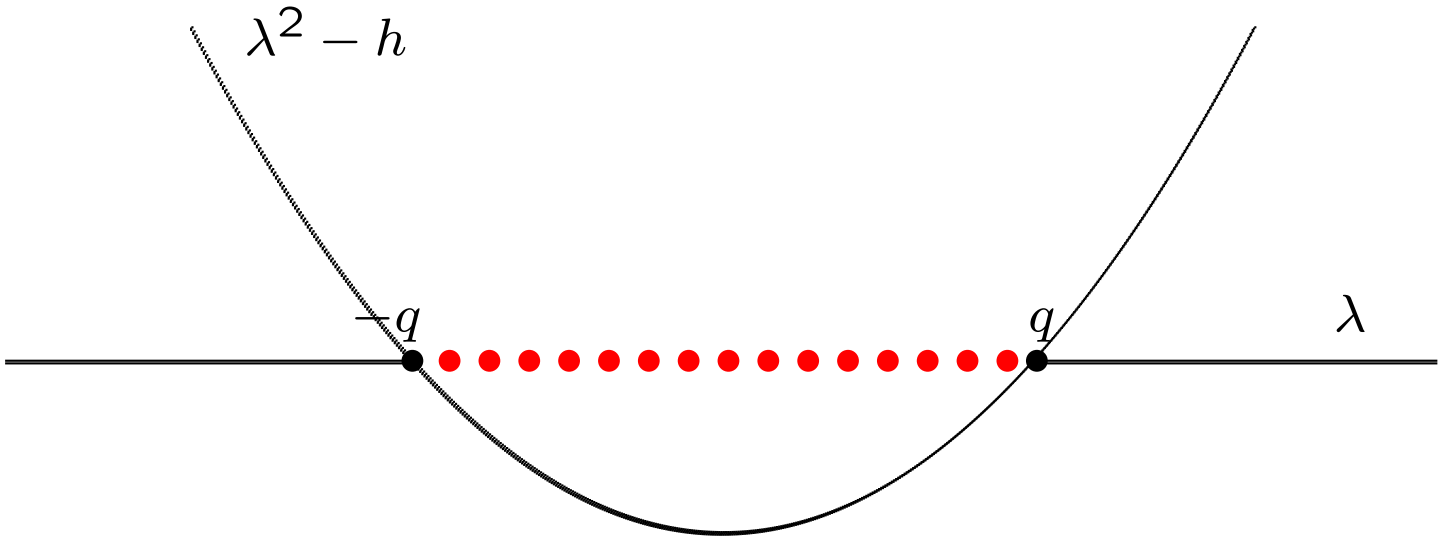
The system of Bethe equations

$$e^{iL\lambda_j} = (-1)^{N-1}, \qquad \lambda_j = \frac{2\pi}{L} I_j$$

$$I_j \neq I_k, \qquad j, k = 1, \dots, N$$

Ground state



$$E = \sum_{j=1}^{N} (\lambda_j^2 - h), \qquad h = q^2, \qquad \lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right)$$

$$N/L \equiv D = q/\pi$$

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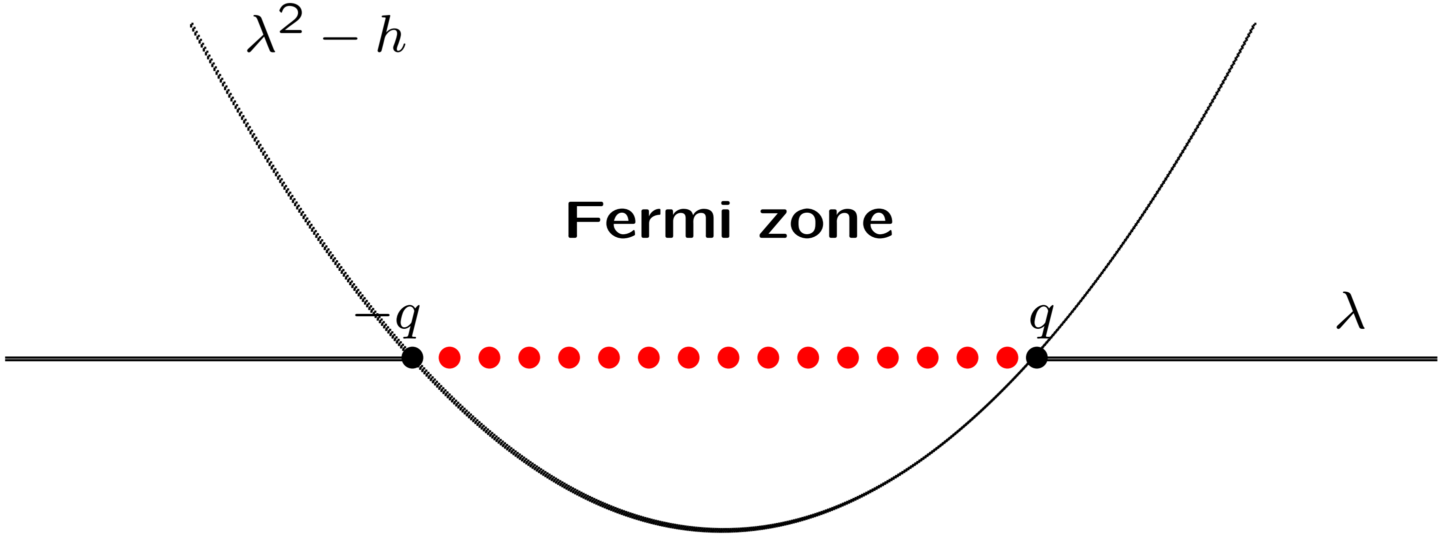

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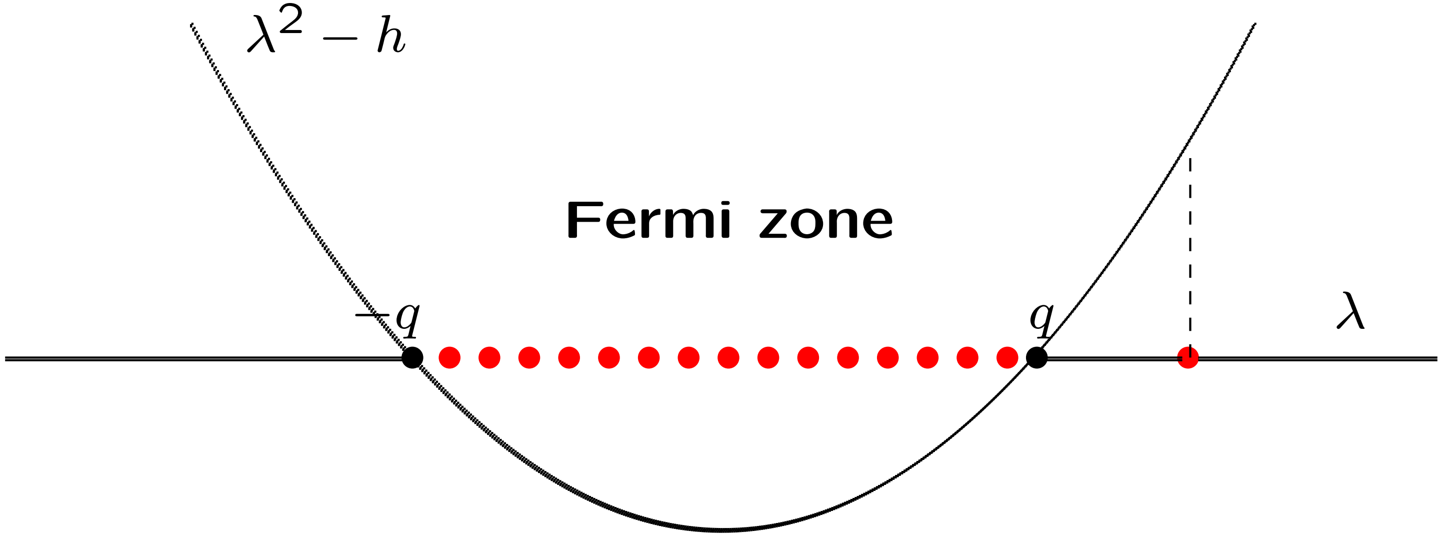

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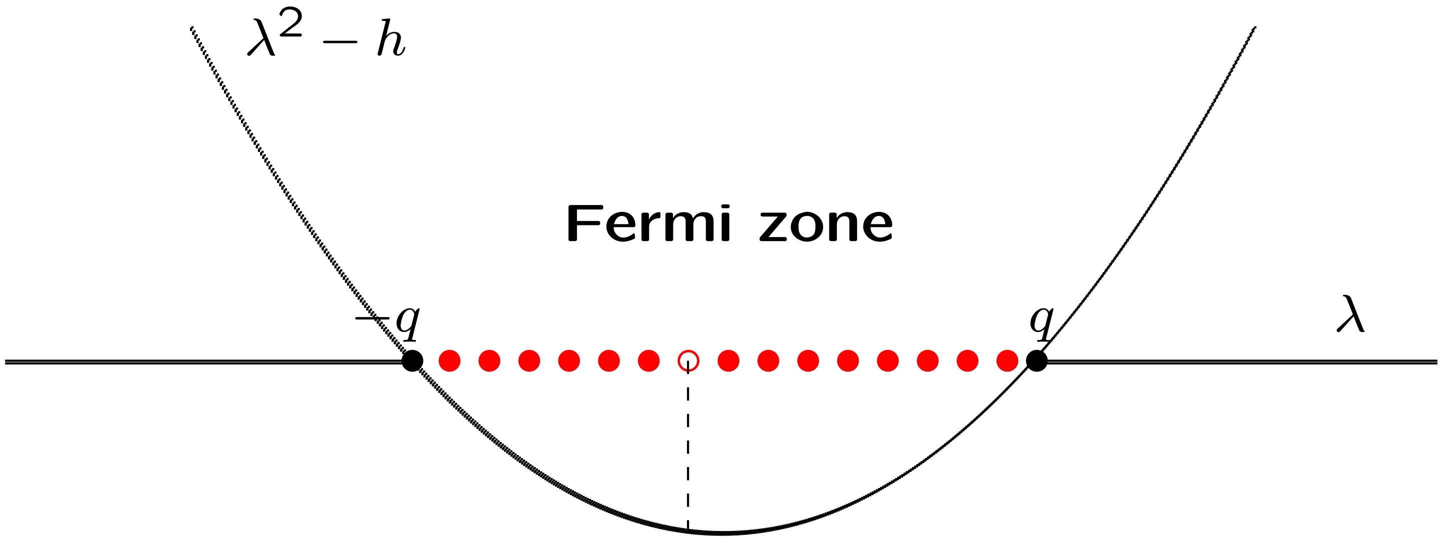

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Particles and holes

$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right) \text{ — ground state}$$

$$\mu_j = \frac{2\pi I_j}{L} \text{ — excited state}$$

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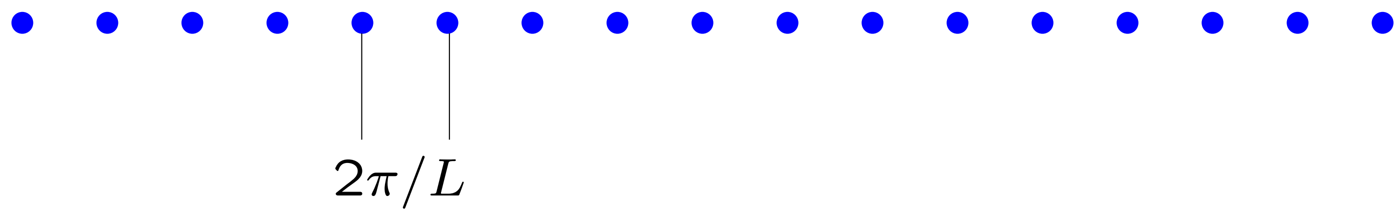
Excited state



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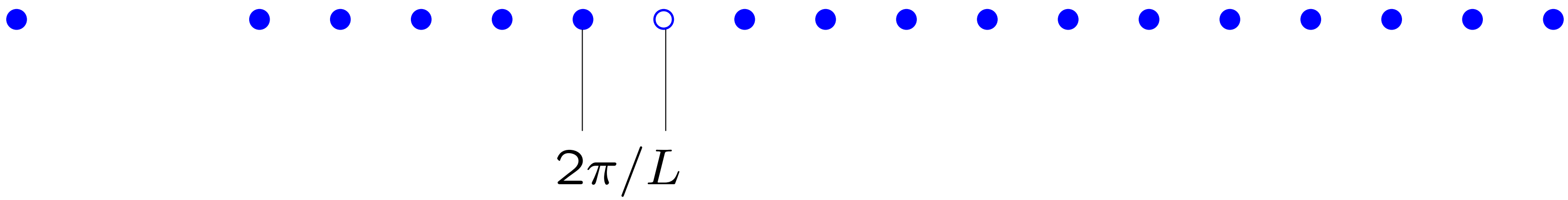

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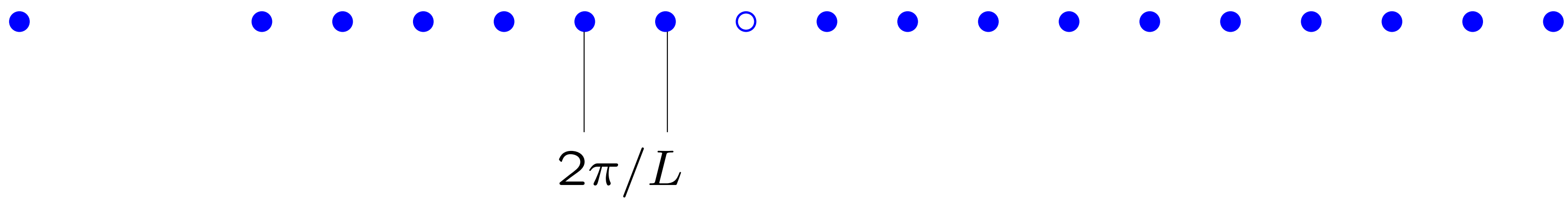

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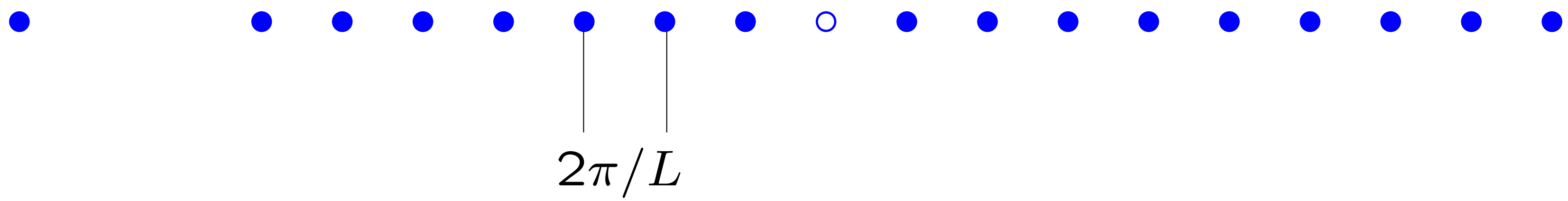

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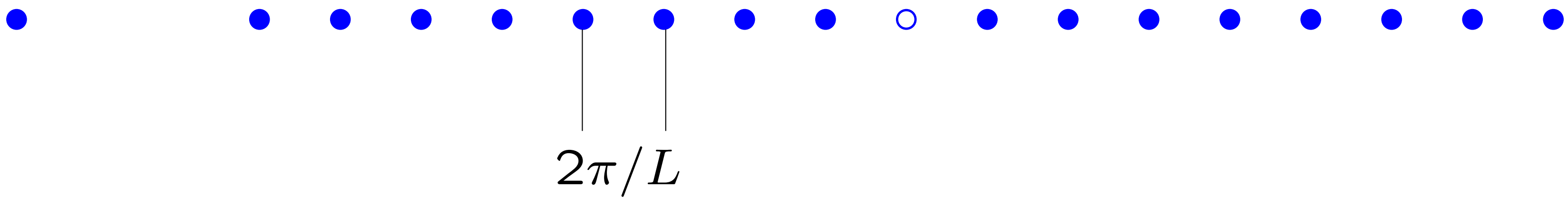

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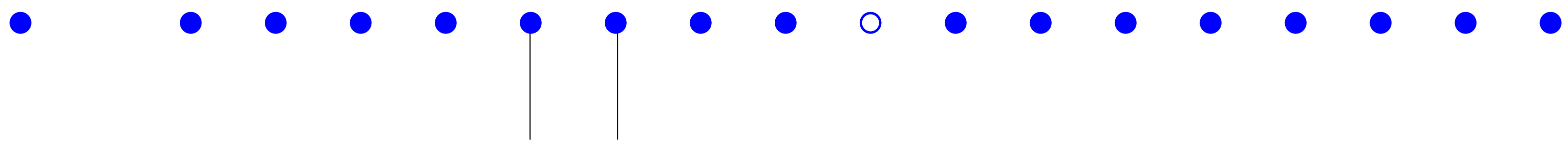

$2\pi/L$

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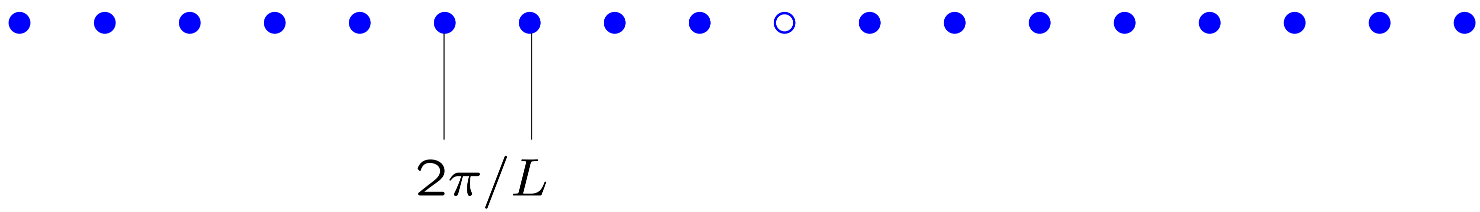

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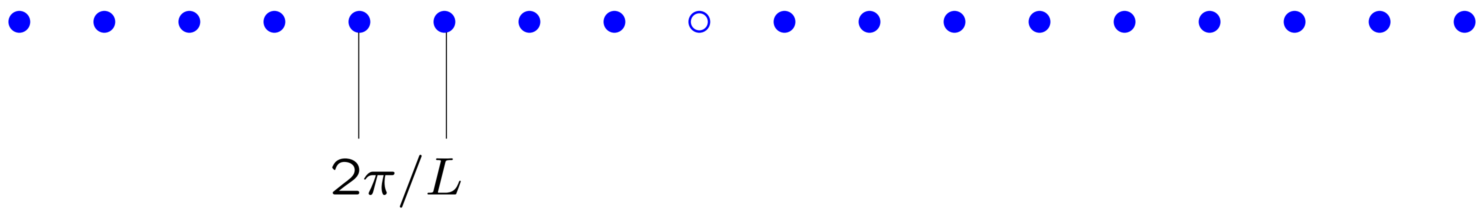

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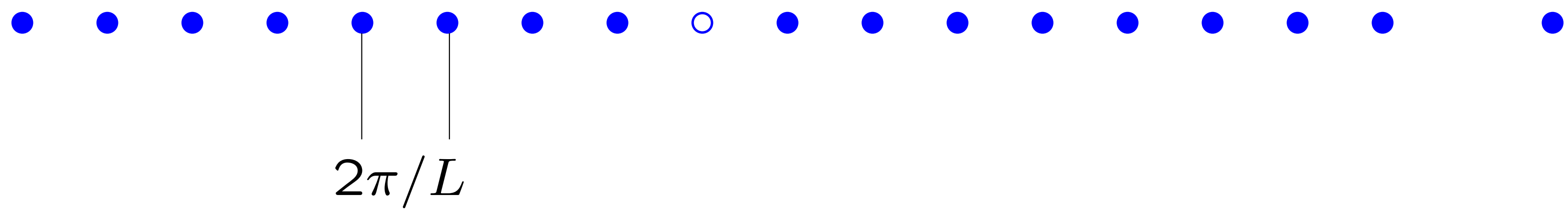

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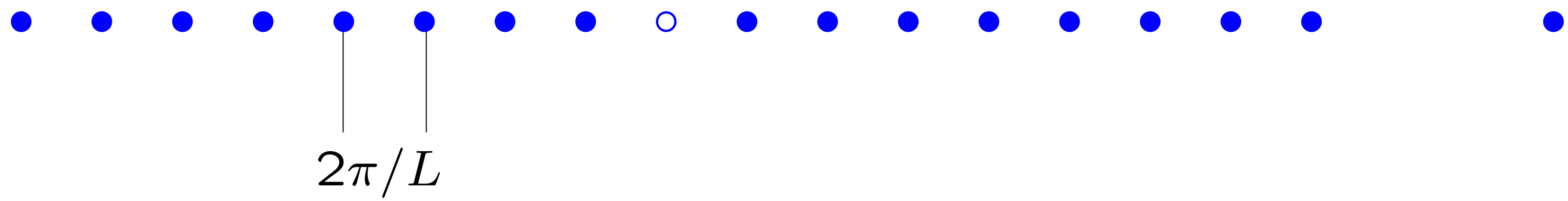

Field form factor

$$\mathcal{F}^{\phi}_{\psi,\psi'} = \frac{\langle \psi | \phi(0,0) | \psi' \rangle}{\| \psi \| \ \| \psi' \|}$$

$$|\psi'\rangle = |\psi'(\mu_1, \dots \mu_{N+1}, L)\rangle \qquad \langle \psi | = \langle \psi(\lambda_1, \dots \lambda_N, L)|$$

$$\left| \mathcal{F}^{\phi}_{\psi,\psi'} \right|^2 = \frac{1}{L} \left(\frac{2}{L} \right)^{2N} \frac{\prod_{j>k}^{N} (\lambda_j - \lambda_k)^2 \prod_{j>k}^{N+1} (\mu_j - \mu_k)^2}{\prod_{j=1}^{N} \prod_{k=1}^{N+1} (\lambda_j - \mu_k)^2}$$

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$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right)$$

$$\mu_j = \frac{2\pi}{L}\left(j - \frac{N+2}{2}\right)$$

$$\left|\mathcal{F}^{\phi}_{\psi,\psi'}\right|^2 = \frac{1}{L}\left(\frac{2}{L}\right)^{2N} \frac{\prod_{j>k}^{N} (\lambda_j - \lambda_k)^2 \prod_{j>k}^{N+1} (\mu_j - \mu_k)^2}{\prod_{j=1}^{N} \prod_{k=1}^{N+1} (\lambda_j - \mu_k)^2}$$

Field form factor

$$\mathcal{F}^{\phi}_{\psi,\psi'} = \frac{\langle\psi|\phi(0,0)|\psi'\rangle}{\|\psi\|\;\|\psi'\|}$$

$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right)$$

$$\mu_j = \frac{2\pi}{L}\left(j - \frac{N+2}{2}\right)$$

$$\left|\mathcal{F}^{\phi}_{\psi,\psi'}\right|^2 = \frac{\pi^2 G^4(1/2)}{L}\,\frac{G^2(N+1)G^2(N+2)}{G^4(N+3/2)}$$

$G(z)$ is Barnes function: $G(z+1) = \Gamma(z)G(z)$

Field form factor

$$\mathcal{F}^{\phi}_{\psi,\psi'} = \frac{\langle\psi|\phi(0,0)|\psi'\rangle}{\|\psi\|\;\|\psi'\|}$$

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$$\left|\mathcal{F}^{\phi}_{\psi,\psi'}\right|^2 = \frac{\pi^2 G^4(1/2)}{L}\,\frac{G^2(N+1)G^2(N+2)}{G^4(N+3/2)} = C\cdot L^{-1/2}$$

$$L, N \to \infty, \qquad N/L = D$$

1-particle 1-hole form factor

Excited state

$$\mu_j = \frac{2\pi}{L}\left(j - \frac{N+2}{2}\right), \qquad j \neq h$$

$$\mu_h = \frac{2\pi}{L}\left(p - \frac{N+2}{2}\right), \qquad p > N+1$$

$$\left|\mathcal{F}^{\phi}_{\psi,\psi'}\right|^2 = \frac{C \cdot L^{-1/2}}{(p-h)^2} \frac{\Gamma^2(p)\Gamma^2(p-N-\frac{1}{2})}{\Gamma^2(p-\frac{1}{2})\Gamma^2(p-N-1)} \cdot \frac{\Gamma^2(h-\frac{1}{2})\Gamma^2(N-h+\frac{3}{2})}{\Gamma^2(h)\Gamma^2(N-h+2)}$$

Contribution of 1-particle 1-hole form factors to the correlation function in the thermodynamic limit

Excited state

$$\sum_{|\psi'\rangle} e^{ix\mathcal{P}-it\mathcal{E}} |\mathcal{F}^{\phi}_{\psi,\psi'}|^2 \to \frac{C \cdot L^{-1/2}}{(2\pi)^2} \int\limits_{[-q,q]} d\mu_h \int\limits_{\mathbb{R}\setminus[-q,q]} d\mu_p \, e^{ix\mathcal{P}-it\mathcal{E}}$$

$$\times \frac{1}{(\mu_p-\mu_h)^2} \left(\frac{\mu_p^2-q^2}{q^2-\mu_h^2}\right)$$

1-particle 1-hole form factor

Excited state

$$\left|\mathcal{F}^{\phi}_{\psi,\psi'}\right|^2 = \frac{C\cdot L^{-1/2}}{(p-h)^2}\frac{\Gamma^2(p)\Gamma^2(p-N-\frac{1}{2})}{\Gamma^2(p-\frac{1}{2})\Gamma^2(p-N-1)}\cdot\frac{\Gamma^2(h-\frac{1}{2})\Gamma^2(N-h+\frac{3}{2})}{\Gamma^2(h)\Gamma^2(N-h+2)}$$

$$\frac{\Gamma^2(h-\frac{1}{2})\Gamma^2(N-h+\frac{3}{2})}{\Gamma^2(h)\Gamma^2(N+2-h)} \to \frac{1}{q^2-\mu_h^2}, \qquad 1 \ll h \ll N$$

1-particle 1-hole form factor

Excited state

$$\left|\mathcal{F}^{\phi}_{\psi,\psi'}\right|^2 = \frac{C\cdot L^{-1/2}}{(p-h)^2}\frac{\Gamma^2(p)\Gamma^2(p-N-\frac{1}{2})}{\Gamma^2(p-\frac{1}{2})\Gamma^2(p-N-1)}\cdot\frac{\Gamma^2(h-\frac{1}{2})\Gamma^2(N-h+\frac{3}{2})}{\Gamma^2(h)\Gamma^2(N-h+2)}$$

$$\frac{\Gamma^2(h-\frac{1}{2})\Gamma^2(N-h+\frac{3}{2})}{\Gamma^2(h)\Gamma^2(N+2-h)} \to \frac{1}{q^2-\mu_h^2}, \qquad 1 \ll h \ll N$$

$$\frac{\Gamma^2(h-\frac{1}{2})\Gamma^2(N-h+\frac{3}{2})}{\Gamma^2(h)\Gamma^2(N+2-h)} \to \frac{\Gamma^2(h-\frac{1}{2})}{\Gamma^2(h)}\cdot N^{-1/2}, \qquad h \ll N$$

Particle-hole form factors have no uniform thermodynamic limit. One should consider separately:

- **particle (hole) is far from the Fermi boundaries**

$$\lim \mu_p \neq \pm q \qquad\qquad \lim \mu_h \neq \pm q$$

- **particle (hole) is close to the Fermi boundaries**

$$\lim \mu_p = \pm q \qquad\qquad \lim \mu_h = \pm q$$

Classes of excited states

Two excited states belong to the same class $\mathbf{P}$ if:

- they have the same excitation momentum $\mathcal{P}$ and energy $\mathcal{E}$ in the thermodynamic limit;

- they have the same number of particles and holes separated from the Fermi boundaries with the same rapidities μ_{p_a} and μ_{h_a} in the thermodynamic limit.

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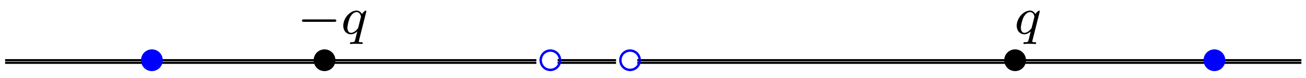
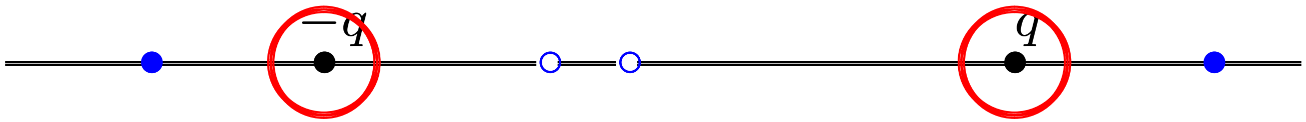
The total number of excitations is not fixed!
One can add arbitrary number of excitations at the Fermi boundaries.

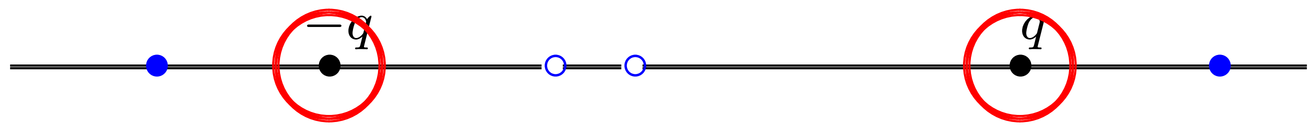
Classes of excited states



Classes of excited states



Classes of excited states

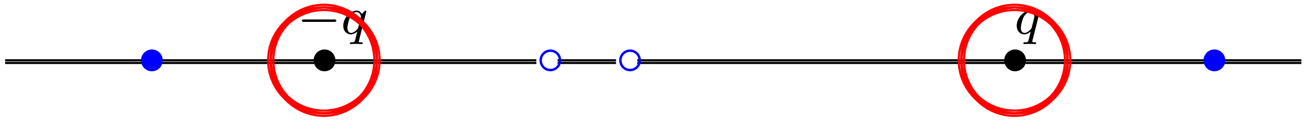


$$n_p(-q) = 0, \quad n_h(-q) = 0 \qquad\qquad n_p(q) = 0, \quad n_h(q) = 0$$

$$\ell = n_p(q) - n_h(q) = n_h(-q) - n_p(-q) = 0$$

ℓ — particle/hole discrepancy

Classes of excited states

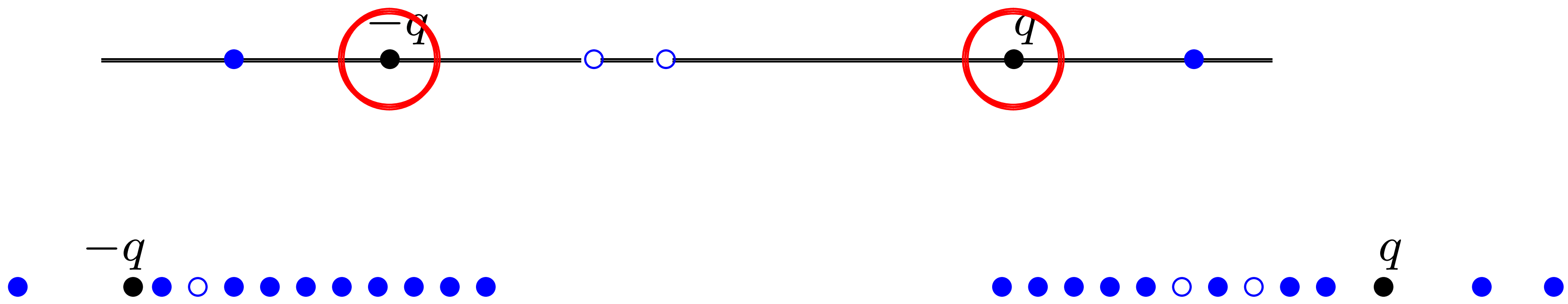


$n_p(-q) = 0, \quad n_h(-q) = 0$ $\qquad$ $n_p(q) = 2, \quad n_h(q) = 2$

$$\ell = n_p(q) - n_h(q) = n_h(-q) - n_p(-q) = 0$$

ℓ — particle/hole discrepancy

Classes of excited states

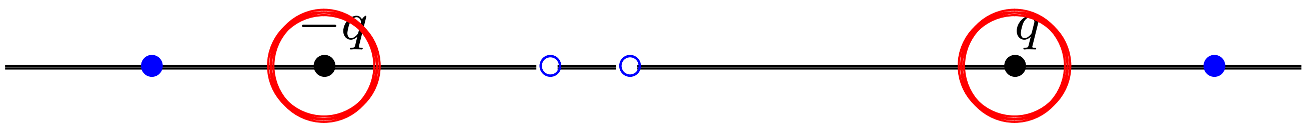


$n_p(-q) = 1, \quad n_h(-q) = 1$ $\qquad\qquad$ $n_p(q) = 2, \quad n_h(q) = 2$

$$\ell = n_p(q) - n_h(q) = n_h(-q) - n_p(-q) = 0$$

ℓ — particle/hole discrepancy

Classes of excited states

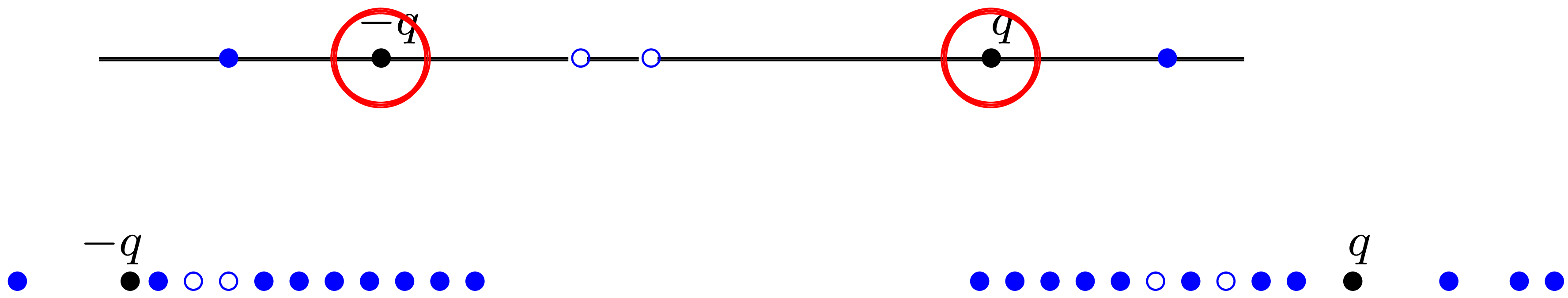


$$n_p(-q) = 0, \quad n_h(-q) = 1 \qquad\qquad n_p(q) = 1, \quad n_h(q) = 0$$

$$\ell = n_p(q) - n_h(q) = n_h(-q) - n_p(-q) = 1$$

ℓ — particle/hole discrepancy

Classes of excited states

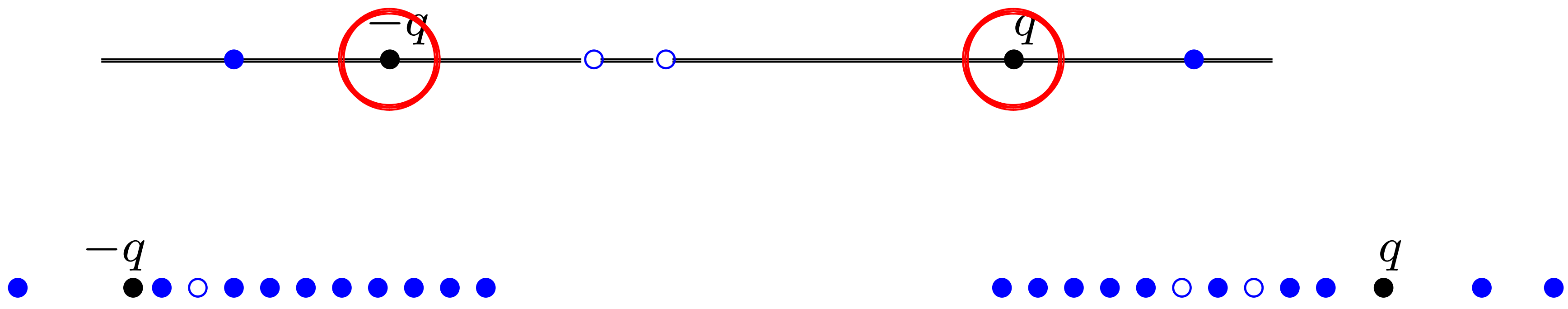


$n_p(-q) = 1, \quad n_h(-q) = 2$ $\qquad$ $n_p(q) = 3, \quad n_h(q) = 2$

$$\ell = n_p(q) - n_h(q) = n_h(-q) - n_p(-q) = 1$$

ℓ — particle/hole discrepancy

Summation within a class of excited states



Original excited state is dressed by a cloud of excitations in vicinities of $\pm q$. As the result original bare form factors turns into dressed form factors.

$$\sum_{|\psi'\rangle\in\mathbf{P}} e^{ix\mathcal{P}-it\mathcal{E}}|\mathcal{F}^{\phi}_{\psi,\psi'}|^2 \rightarrow e^{ix\mathcal{P}-it\mathcal{E}}|\mathcal{F}^{\phi}_{\mathbf{P}}|^2$$

Summation within a class of excited states

$$|\mathcal{F}_{\mathbf{P}}^{\phi}|^2 = L^{-1/2} C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\}) R_{+}(\nu) R_{-}(\nu)$$

Summation within a class of excited states

$$|\mathcal{F}^{\phi}_{\mathbf{P}}|^2 = L^{-1/2} C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\}) R_+(\nu) R_-(\nu)$$

$$R_{\pm}(\nu) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \sum_{\substack{p_1,\ldots,p_n=1 \\ h_1,\ldots,h_n=1}}^{\infty} \left(\det_n \frac{1}{p_j + h_k - 1} \right)^2 \prod_{k=1}^{n} z^{p_k + h_k - 1}$$

$$\times \left(\frac{\sin \pi\nu}{\pi} \right)^{2n} \prod_{k=1}^{n} \frac{\Gamma^2(p_k \pm \nu)\Gamma^2(h_k \mp \nu)}{\Gamma^2(p_k)\Gamma^2(h_k)}$$

$$\nu = 1/2, \qquad z = e^{-2\pi i(2qt \mp x)/L}, \qquad (t \to t - i0)$$

Summation within a class of excited states

$$|\mathcal{F}_{\mathbf{P}}^{\phi}|^2 = L^{-1/2} C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\}) R_+(\nu) R_-(\nu)$$

$$R_\pm(\nu) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \sum_{\substack{p_1,\dots,p_n=1 \\ h_1,\dots,h_n=1}}^{\infty} \left(\det_n \frac{1}{p_j + h_k - 1} \right)^2 \prod_{k=1}^{n} z^{p_k+h_k-1}$$

$$\times \left(\frac{\sin \pi\nu}{\pi} \right)^{2n} \prod_{k=1}^{n} \frac{\Gamma^2(p_k \pm \nu)\Gamma^2(h_k \mp \nu)}{\Gamma^2(p_k)\Gamma^2(h_k)}$$

The sum $R_\pm(\nu)$ was studied in the works of A. Borodin, S. Kerov, A. Okounkov, G. Olshoanski ($\mathbf{Z}$-measures on partitions).

Summation within a class of excited states

$$|\mathcal{F}^{\phi}_{\mathbf{P}}|^2 = L^{-1/2} C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\}) R_+(\nu) R_-(\nu)$$

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$$\times \left(\frac{\sin \pi\nu}{\pi}\right)^{2n} \prod_{k=1}^{n} \frac{\Gamma^2(p_k \pm \nu)\Gamma^2(h_k \mp \nu)}{\Gamma^2(p_k)\Gamma^2(h_k)}$$

$$R_\pm(\nu) = (1-z)^{-\nu^2}$$

Summation within a class of excited states

$$|\mathcal{F}_{\mathbf{P}}^{\phi}|^2 = L^{-1/2} C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\}) R_+(\nu) R_-(\nu)$$

$$|\mathcal{F}_{\mathbf{P}}^{\phi}|^2 = \frac{C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\})}{L^{1/2}\left(1 - e^{-2\pi i(2qt - x)/L}\right)^{1/4}\left(1 - e^{-2\pi i(2qt + x)/L}\right)^{1/4}}$$

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$$|\mathcal{F}^{\phi}_{\mathbf{P}}|^2 = \frac{C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\})}{\sqrt{2\pi}(x-2qt)^{1/4}(x+2qt)^{1/4}}, \qquad L \to \infty$$

Finite coupling ($c < \infty$)

Bethe equations

$$e^{iL\lambda_j} = - \prod_{k=1}^{N} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}, \qquad e^{iL\mu_j} = - \prod_{k=1}^{N+1} \frac{\mu_j - \mu_k + ic}{\mu_j - \mu_k - ic}$$

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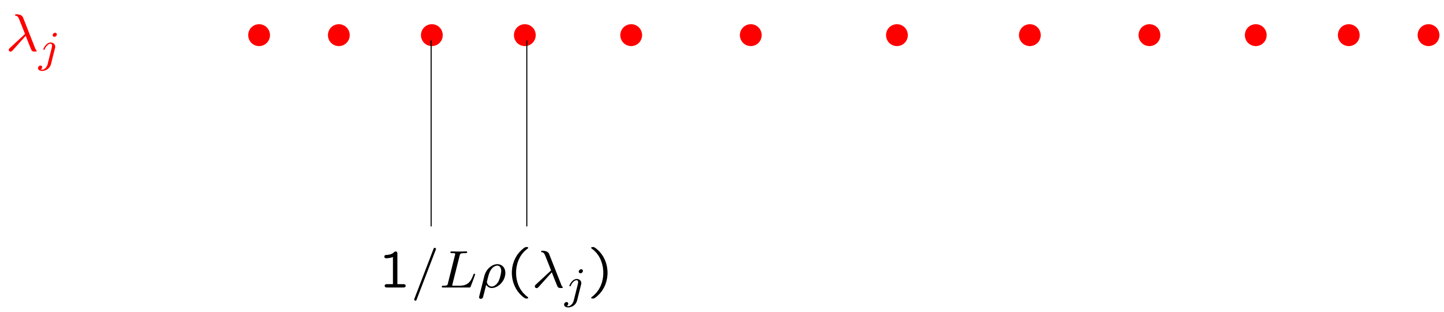

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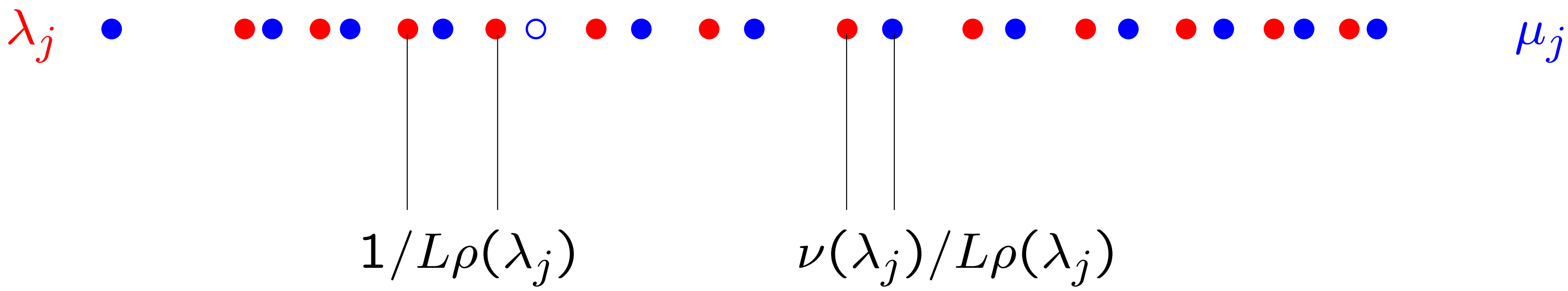
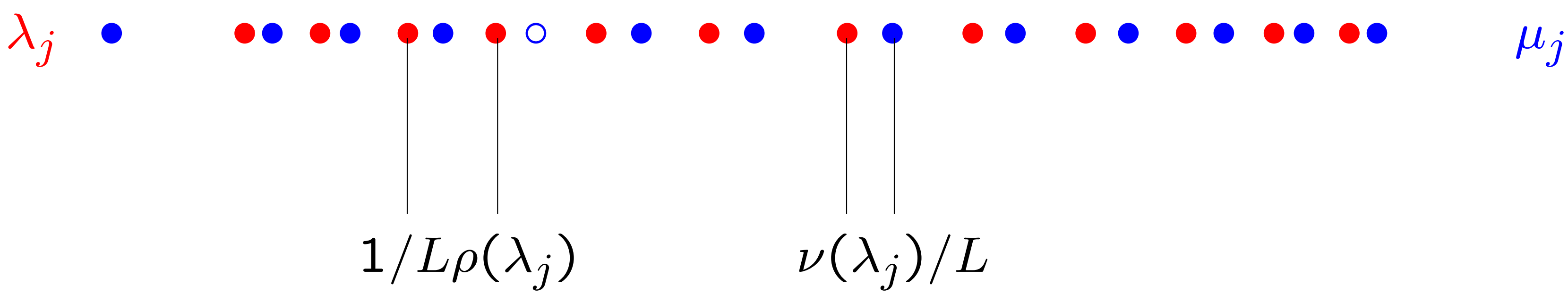


$\nu(\lambda)$ and $\rho(\lambda)$ solve linear integral equations in the $L \to \infty$ limit.

Finite coupling ($c < \infty$)

Form factors

$$|\mathcal{F}^{\phi}_{\psi,\psi'}|^2 = L^{-\nu^2(q)-\nu^2(-q)} C(\{\mu_p\};\{\mu_h\}), \qquad L \to \infty$$



Summation within a class of excited states

$$|\mathcal{F}_{\mathbf{P}}^{\phi}|^2 = L^{-\nu^2(q)-\nu^2(-q)} C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\}) R_+(\nu(q)) R_-(\nu(-q))$$

$$R_{\pm}(\nu) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \sum_{\substack{p_1,\dots,p_n=1 \\ h_1,\dots,h_n=1}}^{\infty} \left(\det_n \frac{1}{p_j + h_k - 1} \right)^2 \prod_{k=1}^{n} z^{p_k+h_k-1}$$

$$\times \left(\frac{\sin \pi\nu}{\pi} \right)^{2n} \prod_{k=1}^{n} \frac{\Gamma^2(p_k \pm \nu)\Gamma^2(h_k \mp \nu)}{\Gamma^2(p_k)\Gamma^2(h_k)}$$

$$z = \exp\left(-\frac{2\pi i}{L}(vt \mp x) \right), \qquad v \text{ is a constant}$$

Summation within a class of excited states

$$|\mathcal{F}^{\phi}_{\mathbf{P}}|^2 = L^{-\nu^2(q)-\nu^2(-q)} C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\}) R_+(\nu(q)) R_-(\nu(-q))$$

$$|\mathcal{F}^{\phi}_{\mathbf{P}}|^2 = \frac{C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\})}{L^{\nu^2(q)+\nu^2(-q)} \left(1 - e^{-2\pi i(vt-x)/L}\right)^{\nu^2(q)} \left(1 - e^{-2\pi i(vt+x)/L}\right)^{\nu^2(-q)}}$$

$$|\mathcal{F}^{\phi}_{\mathbf{P}}|^2 = \frac{C_{\mathbf{P}}(\{\mu_p\};\{\mu_h\}) e^{\frac{i\pi}{2}(\nu^2(q)-\nu^2(-q))}}{\Big(2\pi(x-vt)\Big)^{\nu^2(q)} \Big(2\pi(x+vt)\Big)^{\nu^2(-q)}}, \qquad L \to \infty$$

$$G^{\phi}(x,t) = \sum_{\mathbf{P}} e^{ix\mathcal{P}-it\mathcal{E}} |\mathcal{F}^{\phi}_{\mathbf{P}}|^2$$

The sum over different classes $\mathbf{P}$ means:

- The sum over the numbers of particles (N_p) and holes (N_h) separated from the Fermi boundaries;
- The sum over the particle/hole discrepancy (ℓ);
- Integration over particle and hole rapidities μ_p and μ_h.

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- The sum over the particle/hole discrepancy (ℓ);

- Integration over particle and hole rapidities μ_p and μ_h.

Generically analytical evaluation of the integrals over μ_p and μ_h is hardly possible because of very complicated dependence of $\mathcal{F}_{\mathbf{P}}$ on $\{\mu_p\}$ and $\{\mu_h\}$.

- In the asymptotic regime ($x \to \infty$, $t \to \infty$) the integrals over particle/hole rapidities are localized in the vicinities of the Fermi boundaries $\pm q$ and in the vicinity of the saddle point (if the last one exists). In this case we reproduce the CFT prediction for the asymptotics and some additional contributions coming from the saddle point. The last ones are dominant for certain correlation functions.

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- There exists always a possibility to compute the integrals over particle/hole rapidities numerically.

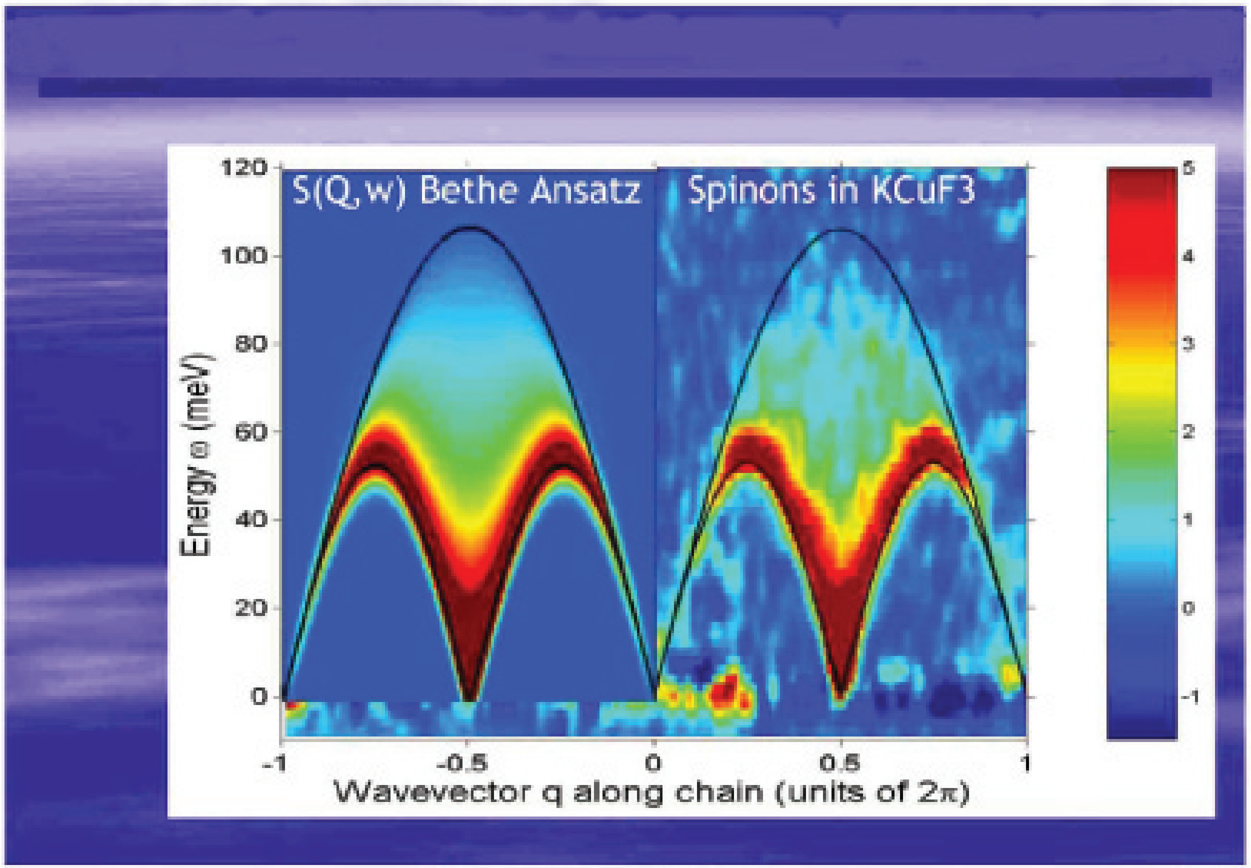
Numerical summation over classes for XXZ chain

$$H = \sum_{k=1}^{L} (S_k^x S_{k+1}^x + S_k^y S_{k+1}^y + \Delta S_k^z S_{k+1}^z)$$

We compute the correlation function of the third components of spin $\langle S_m^z(t) S_1^z(0) \rangle$. Its Fourier transform $S(k, w)$ can be measured experimentally.

$$S(k, w) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} e^{imk - itw} \langle S_m^z(t) S_1^z(0) \rangle \, dt$$

Inelastic neutron scattering in $KCuF_3$



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