LOCAL GEOMETRY OF NONREGULAR CARNOT-CARATHÉODORY SPACES AND APPLICATIONS TO NONLINEAR CONTROL THEORY

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Motivation

Geometric control theory

◇ The linear system of ODE ($x \in \mathbb{M}^N$, $m < N$)

$$\dot{x} = \sum_{i=1}^{m} a_i(t) X_i(x) \tag{1}$$

is locally controllable iff $Lie\{X_1, X_2, \ldots, X_m\} = T\mathbb{M}$, i.e. the "horizontal" distribution $H\mathbb{M} = \{X_1, X_2, \ldots, X_m\}$ is bracket-generating:

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• span$\{X_I(v) : |I| \leq M\} = T_v\mathbb{M}$ for all $v \in \mathbb{M}$, where $X_I = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]$, $|I| = k$ (Hörmander's condition)

• M is the depth of the sub-Riemannian space $\mathbb{M}$

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◇ The linear system of ODE ($x \in \mathbb{M}^N$, $m < N$)

$$\dot{x} = \sum_{i=1}^{m} a_i(t) X_i(x) \tag{3}$$

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• M is the depth of the sub-Riemannian space $\mathbb{M}$

• Rashevsky-Chow theorem $\Rightarrow$ on $\mathbb{M}$ there exists an intrinsic metric $d_c(u,v) = \inf\limits_{\substack{\gamma-\text{horizontal} \\ \gamma(0)=u, \gamma(1)=v}} \{L(\gamma)\}$

• Filtration $H\mathbb{M} = H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M = T\mathbb{M}$ such that

$$[H_1, H_i] = H_{i+1}$$

(Hörmander's condition).

Here

$$H_k(v) = \mathrm{span}\{[X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots](v) : X_{i_j} \in H_1\}$$

• A point $u \in \mathbb{M}$ is called regular if $\dim H_k(v) = \mathrm{const}$ in some neighborhood $v \in U(u) \subseteq \mathbb{M}$.

Otherwise, u is called nonregular.

• Examples. Regular: Heisenberg groups, Carnot groups, roto-translation group, etc.

Nonregular: Groushin-type planes (related to the PDE $\frac{\partial^2 u}{\partial x^2} + x^{2k}\frac{\partial^2 u}{\partial x^2} = f$)

$\mathbb{M} = \mathbb{R}^2$. $H_1 = \mathrm{span}\{X_1 = \frac{\partial}{\partial x},\ X_2 = x^k \frac{\partial}{\partial y}\}$.

The axis $x = 0$ consists of nonregular points; the depth is $M = k + 1$.

There are no regular C-C structures on $\mathbb{R}^2$!

$\diamond$ (J.-M. Coron, etc.) The sufficient condition of controllability of the nonlinear system

$$\begin{cases} \dot{x} = f(x, a), \\ x(0) = x_0, \end{cases} \tag{4}$$

is that

$$\mathrm{span}\Big\{h(0) : h \in \mathrm{Lie}\, \frac{\partial^{|\alpha|}}{\partial a^\alpha} f(0, \cdot), \alpha \in \mathbb{N}^M\Big\} = T_{x_0}\mathbb{M}$$

for some $M \in \mathbb{N}$.

$\diamond$ (J.-M. Coron, etc.) The sufficient condition of controllability of the nonlinear system

$$\begin{cases} \dot{x} = f(x,a), \\ x(0) = x_0, \end{cases} \tag{5}$$

is that

$$\text{span}\Big\{h(0) : h \in \text{Lie}\,\frac{\partial^{|\alpha|}}{\partial a^\alpha} f(0,\cdot), \alpha \in \mathbb{N}^M\Big\} = T_{x_0}\mathbb{M}$$

for some $M \in \mathbb{N}$. Letting

$$F_\nu = \Big\{\frac{\partial^\alpha}{\partial a^\alpha} f(0,\cdot) : |\alpha| \le \nu\Big\}$$

and

$$H_k(q) = \text{span}\{[X_1,[X_2,\ldots,[X_{i-1},X_i]\ldots](q) : X_j \in F_{\nu_j}, \nu_1+\nu_2+\ldots+\nu_i \le k\},$$

one obtains a weighted filtration

$$H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M = T\mathbb{M}, \text{ such that } [H_i, H_j] \subseteq H_{i+j}$$

more general than the Hörmander condition

Some references concerning the underlying geometry

- Nagel, Stein, Wainger 1985;
- Gromov 1996;
- Coron 1996;
- Christ, Nagel, Stein, Wainger 1999;
- Rampazzo, Sussmann 2001, 2007
- Tao, Wright 2003
- Agrachev, Marigo 2003;
- Montanari, Morbidelli 2004, 2011;
- Street 2011
- Karmanova, Vodopyanov 2007–2009; Karmanova 2010, 2011.

Weighted Carnot-Carathéodory spaces

- $\mathbb{M}$, $\dim \mathbb{M} = N$ is a smooth connected manifold

- $X_1, X_2, \ldots, X_q \in C^{2M+1}$ span $T\mathbb{M}$; $\deg X_i := d_i$, $d_1 \le \ldots \le d_q$.

- $X_I = [X_{i_1}, [\ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]$, where $I = (i_1, \ldots, i_k)$;
$|I|_h := d_{i_1} + \ldots + d_{i_k}$.

- $H_j = \mathrm{span}\{X_I \mid |I|_h \le j\}$.

$$H\mathbb{M} = H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M = T\mathbb{M}$$

$$[H_i, H_j] \subseteq H_{i+j}.$$

Here $[H_i, H_j]$ is the linear span of commutators of the vector field generating H_i and H_j.
Model case: $d_1 := 1$, $d_q := M$.

Problems

1. In a neighborhood of a nonregular point, the basis $Y_1, Y_2, \ldots, Y_N$, associated to the filtration $H_1 \subseteq H_2 \subseteq \ldots \subseteq H_M$, varies discontinuously from point to point.

2. In the case of a weighted filtration the intrinsic Carnot-Carathéodory metric d_c might not exist.

Example (Stein "Harmonic Analysis")

$\mathbb{M} = \mathbb{R}^N$ with standard basis $\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N}$.

Let deg$(\partial_{x_i}) = 1$ for $1 \leq i \leq m$; deg$(\partial_{x_i}) > 1$ for $i > m$.

Evidently, $H_i = \text{span}\{\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_i}\}$ satisfy $[H_i, H_j] \subseteq H_{i+j}$, since $[H_i, H_j] = \{0\}$.

But $H_1 = \text{span}\{\partial_{x_i}\}_{i=1}^m$ (for any $m < N$) does not span $\mathbb{R}^N$.

3. Different choices of weights may lead to different combinations of regular and nonregular points.

Example

$\mathbb{M} = \mathbb{R}^3$; vector fields $\{X_1 = \partial_y, X_2 = \partial_x + y\partial_t, X_3 = \partial_x\}$.

Nontrivial commutator: $[X_1, X_2] = \partial_t$.

1. Let $\deg(X_i) := 1$, $i = 1, 2, 3$. Then $\deg([X_1, X_2]) = 2$ and

$$H_1 = \text{span}\{X_1, X_2, X_3\}, \; H_2 = H_1 \cup \text{span}\{[X_1, X_2]\}.$$

In this case $\{y = 0\}$ is a plane consisting of nonregular points.

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In this case $\{y = 0\}$ is a plane consisting of nonregular points.

2. Let $\deg(X_1) := a$, $\deg(X_2) := b$, $\deg(X_3) := a + b$, $a \le b$.
Then $\deg([X_1, X_2]) = a + b \ \Rightarrow$

$H_a = \mathrm{span}\{X_1\}$, $H_b = H_a \cup \mathrm{span}\{X_2\}$, $H_{a+b} = H_a \cup H_b \cup \mathrm{span}\{X_3, [X_1, X_2]\}$

In this case all points of $\mathbb{R}^3$ are regular.

Questions

Are there some analogs of classical results of sub-Riemannian geometry for weighted C-C spaces?

$\diamond$ Results on existence and the algebraic structure of the Gromov's tangent cone to $M = (\mathbb{M}, d_c)$ at a fixed point $u \in \mathbb{M}$: it is a homogeneous space of a Carnot group $(G/H, d_c^u)$.

$\diamond$ Local approximation theorem: if $d_c(u, v) = O(\varepsilon)$ and $d_c(u, v) = O(\varepsilon)$, then $|d_c(u, v) - d_c^u(v, w)| = O(\varepsilon^{1+\frac{1}{M}})$.

$\diamond$ Methods of optimal motion planning for the system (1).

Metric structure

We work with the following quasimetric **Nagel, Stein, Wainger 1985:**

$$\rho(v,w) = \mathrm{inf}\{\delta > 0 \mid \text{ there is a curve } \gamma : [0,1] \to U \ ,$$

$$\gamma(0) = v, \gamma(1) = w, \dot{\gamma}(t) = \sum_{|I|_h \leq M} w_I X_I(\gamma(t)), |w_I| < \delta^{|I|_h}\}.$$

Here $X_I = [X_{i_1}, [\ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]$, where $I = (i_1, \ldots, i_k)$;
$|I|_h = d_{i_1} + \ldots + d_{i_k}$.

For the regular case $\rho(v,w) = d\infty(v,w) = \max_{i=1,\ldots,N} \{|v_i|^{\frac{1}{\deg Y_i}}\}$

Quasimetric space (X, d_X)

X is a topoogical space; $d_X : X \times X \to \mathbb{R}^+$ is such that

(1) $d_X(u,v) \geq 0$; $d_X(u,v) = 0 \Leftrightarrow u = v$;

(2) $d_X(u,v) \leq c_X d_X(v,u)$, where $1 \leq c_X < \infty$ uniformly on $u, v \in X$ (generalized symmetry property);

(3) $d_X(u,v) \leq Q_X(d_X(u,w) + d_X(w,v))$, where $1 \leq Q_X < \infty$ uniformly on all $u, v, w \in X$ (generalized triangle inequality);

(4) $d_X(u,v)$ upper semicontinuous on the first argument

$Q_X = c_X = 1 \Rightarrow (X, d_X)$ **metric space**

Show picture

Basic considerations

- Choice of basis $\{Y_1, Y_2, \ldots, Y_N\}$ among $\{X_I\}_{|I|_h \leq M}$:

$*$ $Y_1, Y_2, \ldots, Y_N$ are linearly independent at u (hence in some neighborhood $U(u)$);

$*$ $\sum\limits_{i=1}^{N} \deg Y_i$ is minimal;

$*$ $\sum\limits_{j=1}^{N} |I_j|$ is minimal, where $Y_j = X_{I_j}$.

- Coordinates of the second kind $\Phi^u : \mathbb{R}^N \to U$

$$\Phi^u(x_1, \ldots, x_N) = \exp(x_1 Y_1) \circ \exp(x_2 Y_2) \circ \ldots \circ \exp(x_N Y_N)(u)$$

Basic considerations

- $\{\widehat{X}_I^u\}_{|I|_h \le M}$ – nilpotent approximations of $\{X_I\}_{|I|_h \le M}$ at $u \in U$.

$H_j(u) = \widehat{H}_j(u)$, where $H_j = \mathrm{span}\{\widehat{X}_I^u\}_{|I|_h \le j}$, $\widehat{H}_j = \mathrm{span}\{\widehat{X}_I^u\}_{|I|_h \le j}$.

- Quasimetic

$$\rho^u(v,w) = \inf\{\delta > 0 \mid \text{ there is a curve } \gamma : [0,1] \to U,$$

$$\gamma(0) = v, \gamma(1) = w, \dot{\gamma}(t) = \sum_{|I|_h \le M} w_I \widehat{X}_I^u(\gamma(t)), |w_I| < \delta^{|I|_h}\}.$$

Conical property:

$$\rho^u(\Delta_\varepsilon^u v, \Delta_\varepsilon^u w) = \varepsilon \rho^u(v,w)$$

where Δ_ε^u are dilations induced by the homogeneous weight structure.

Divergence of integral lines

Let $u, v \in U$, $r > 0$. *Divergence of integral lines* with the center of nilpotentization u on $B(v,r)$ is

$$R(u,v,r) = \max\{ \sup_{\widehat{y} \in B^{\rho^u}(v,r)} \{\rho^u(y,\widehat{y})\}, \sup_{y \in B^{\rho}(v,r)} \{\rho(y,\widehat{y})\}\} \qquad (6)$$

Here the points y and $\widehat{y}$ are defined as follows. Let $\gamma(t)$ be an arbitrary curve such that

$$\begin{cases} \dot{\gamma}(t) = \sum\limits_{|I|_h \le M} b_I \widehat{X}_I^u(\gamma(t)), \\ \gamma(0) = v, \gamma(1) = \widehat{y}, \end{cases}$$

and

$$\rho^u(v,\widehat{y}) \le \max_{|I|_h \le M} \{|b_I|^{1/|I|_h}\} \le r.$$

$y = \exp(\sum\limits_{|I|_h \le M} b_I \widehat{X}_I^u)(v)$. So sup in (6) is taken over infinite set of points $\widehat{y} \in B^{\rho^u}(v,r)$ and reals $\{b_I\}_{|I|_h \le M}$,

Main result

Theorem 1 (Estimate of divergence of integral lines).

Let $u, v \in U$, $\rho(u,v) = O(\varepsilon)$, $r = O(\varepsilon)$ and $B^{\rho}(v,r) \cup B^{\rho^u}(v,r) \subseteq U$. Then the following estimate on the divergence of integral lines holds: $R(u,v,r) = O(\varepsilon^{1+\frac{1}{M}})$.

Can be used for constructing motion planning algorithms for the nonlinear control system (2): $\dot{x} = f(x,a)$.

Corollaries

- **Theorem 2 (Local approximation theorem)**.

If $u, v, w \in U$, $\rho(u,v) = O(\varepsilon)$ and $\rho(u,w) = O(\varepsilon)$, then

$$|\rho(v,w) - \rho^u(v,w)| = O(\varepsilon^{1+\frac{1}{M}}).$$

- **Theorem 3 (Tangent cone theorem).**

The quasimetric space (U, ρ^u) is the tangent cone to the quasimetric space (U, ρ) at $u \in U$; the tangent cone is isomorphic to G/H, where G is a nilpotent graded group.

• New proofs of the classical results for Hörmander vector fields:

$*$ Rashevsky-Chow theorem (existence of d_c);

$*$ Local approximation theorem

$$|d_c(v,w) - d_c^u(v,w)| = O(\varepsilon^{1+\frac{1}{M}});$$

(Gromov 1996, Bellaiche 1996);

$*$ Tangent cone theorem (Mitchell 1985, Gromov 1996, Bellaiche 1996);

$*$ Motion planning algorithms for the linear control system (1) (Jean 2001, etc.).

Methods of proofs

• Theorem on divergence of integral lines for regular C-C spaces (Vodopyanov, Karmanova 2007–2009; Karmanova 2010–2011;

• Study of geometric properties of the quasimetrics ρ and ρ^u (generalized triangle inequalities, “Rolling-of-the-box” lemmas, etc.);

• Generalization and synthesis of the classical methods of embedding a sub-Riemannian manifold into a regular one (Hermes 1991, Bellaiche 1996, Christ, Nagel, Stein, Wainger 1999; Jean 2001).

Metrical aspect

• We introduce a theory o convergence of quasimetric spaces such that

1) For metric spaces, it is equivalent to Gromov's theory;

2) For boundedly compact quasimetric spaces the limit is unique up to isometry;

3) It gives an adequate notion of the tangent cone.

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(4) $d_X(u,v)$ upper semicontinuous on the first argument

Gromov's theory for metric spaces does not work!

We introduce the distance

$$d_{qm}(X,Y) = \text{inf}\{\rho > 0 \mid \exists f : X \to Y, g : Y \to X, \text{such that}$$

$$\max\left\{\text{dis}(f),\ \text{dis}(g),\ \sup_{x\in X} d_X(x, g(f(x))),\ \sup_{y\in Y} d_Y(y, f(g(y)))\right\} \leq \rho\}$$

where $\text{dis}(f) = \sup_{u,v\in X} |d_Y(f(u), f(v)) - d_X(u,v)|$.

Property. For metric spaces d_{qm} is equivalent to d_{GH}:

$$d_{GH}(X,Y) \leq d_{qm}(X,Y) \leq 2d_{GH}(X,Y).$$

• For noncompact quasimetric spaces we say that $(X_n, p_n) \underset{qm}{\to} (X, p)$, if there is such $\delta_n \to 0$, that for all $r > 0$ there exist mappings $f_{n,r} : B^{d_{X_n}}(p_n, r + \delta_n) \to X$, $g_{n,r} : B^{d_X}(p, r + 2\delta_n) \to X_n$ such that

(1) $f_{n,r}(p_n) = p$, $g_{n,r}(p) = p_n$;

(2) $\text{dis}(f_{n,r}) < \delta_n$, $\text{dis}(g_{n,r}) < \delta_n$;

(3) $\sup\limits_{x \in B^{d_{X_n}}(p_n, r+\delta_n)} d_{X_n}(x, g_{n,r}(f_{n,r}(x))) < \delta_n$.

• $T_x X = \lim\limits_{\lambda \to \infty} (X, x, \lambda \cdot d)$ is the tangent cone to X at $x \in X$

For quasimetric spaces with dilations, in particular Carnot-Carathéodory spaces, we can take $f_n = \Delta^x_{\lambda_n}$, $g_n = \Delta^x_{\lambda_n^{-1}}$ where $\lambda \to \infty$, and prove a tangent cone result.

THANK YOU FOR YOUR ATTENTION!