An elementary proof of the vanishing of the second Lie algebra cohomology of the Witt and Virasoro algebra with values in the adjoint module

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Outline

1. deformations of Lie algebras, how they appear in families, moduli problems
2. we deal with one of the most important infinite dimensional Lie algebra, the Witt algebra $\mathcal{W}$ and its universal central extension the Virasoro algebra $\mathcal{V}$
3. deformations of the Lie algebra L are related to Lie algebra cohomology, in more detail $\mathrm{H}^2(L, L)$ governs the infinitesimal and formal deformations.
4. if $\mathrm{H}^2(L, L) = \{0\}$ then L is infinitesimally and formally rigid
5. present an elementary proof of $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = 0$
6. Warning: For $\mathcal{W}$ and $\mathcal{V}$ exist geometric families (of algebras of Krichever-Novikov type) which are locally non-trivial

W a Lie algebra over a field $\mathbb{K}$ with its bracket $[.,.]$
we write it with an anti-symmetric bilinear map

$$\mu_0 : W \times W \to W, \qquad \mu_0(x,y) = [x,y],$$

fulfilling certain additional conditions corresponding to the Jacobi identity.
On the same vector space W is modeled on, we consider a family of Lie structures

$$\mu_s = \mu_0 + s \cdot \psi_1 + s^2 \cdot \psi_2 + \cdots \ ,$$

with bilinear maps $\psi_i : W \times W \to W$ such that $W_s := (W, \mu_s)$ is a Lie algebra and W_0 is the Lie algebra we started with.

The family $\{W_s\}$ is a deformation of W_0.

Question: what is s ?

1. s a variable which allows to plug in numbers $\alpha \in \mathbb{K}$. Then W_α is a Lie algebra for every α for which the expression above is defined.
   deformation over the affine line $\mathbb{K}[s]$ or over the convergent power series $\mathbb{K}\{\{s\}\}$.
   we obtain a geometric or an analytic deformation respectively.
2. s is as a formal variable. It is possible that μ_s does not exist if we plug in for s any value different from 0.
   deformations over the ring of formal power series $\mathbb{K}[[s]]$.
   We obtain a formal deformation.
3. s is a infinitesimal variable, i.e. we have $s^2 = 0$.
   infinitesimal deformations defined over the quotient $\mathbb{K}[X]/(X^2) = \mathbb{K}[[X]]/(X^2)$.

Equivalence of deformation

two families μ_s and μ'_s deforming the same μ_0 are equivalent if there exists a linear automorphism

$$\phi_s = id + s \cdot \alpha_1 + s^2 \cdot \alpha_2 + \cdots$$

with $\alpha_i : W \to W$ linear maps such that

$$\mu'_s(x, y) = \phi_s^{-1}(\mu_s(\phi_s(x), \phi_s(y))).$$

There is always the trivially deformed family given by $\mu_s = \mu_0$ for all s.
(W, μ_0) is called rigid if every deformation μ_s is locally equivalent to the trivial family
intuitively: W cannot be deformed
depends crucially on the nature of the deformation parameter

Recall

$$\mu_s = \mu_0 + s \cdot \psi_1 + s^2 \cdot \psi_2 + \cdots \ ,$$

Jacobi identity says

$$\mu_s(\mu_s(x, y), z) + \text{cycl. perm.} \ = 0$$

Consider this to all orders of s.

s^0: Jacobi identity for W

s^1:

$$\psi_1([x, y], z) + \text{cycl. perm.} \ + [\psi_1(x, y), z] + \text{cycl. perm.} = 0$$

This is a Lie algebra 2-cocycle - with values in the adjoint module

Lie algebra cohomology

An antisymmetric map $\psi : W \times W \to W$ is a Lie algebra two-cocycle with values in the adjoint module if

$$(d_2\psi)(x,y,z) = \psi([x,y],z) + \psi([y,z],x) + \psi([z,x],y) \\ -[x,\psi(y,z)] + [y,\psi(z,x)] - [z,\psi(x,y)] = 0.$$

It is a coboundary if there exists a linear map $\phi : W \to W$ with

$$\psi(x,y) = (d_1\phi)(x,y) = \phi([x,y]) - [x,\phi(y)] + [y,\phi(x)].$$

Second cohomology of W with values in the adjoint representation is

$$\mathrm{H}^2(W,W) = \ker d_2 / \mathrm{i}m\, d_1$$

Similar $\mathrm{H}^2(W,\mathbb{K})$ (values in the trivial module) – related to central extensions of W.

Well-known results:

1. $\mathrm{H}^2(W, W)$ classifies infinitesimal deformations (Gerstenhaber)
2. If $\dim \mathrm{H}^2(W, W) < \infty$, then all formal deformations up to equivalence can be realized in this vector space (Fialowski, Fuks and Fialowski)
3. If $\mathrm{H}^2(W, W) = 0$, then W is infinitesimally and formally rigid
4. If $\dim W < \infty$, then $\mathrm{H}^2(W, W) = 0$ implies that W is also rigid in the geometric and analytic sense (Gerstenhaber, Nijenhus and Richardson)

algebraic realization:
$\mathbb{K}$ a field of $char(\mathbb{K}) = 0$
The Witt algebra $\mathcal{W}$ is the Lie algebra generated as vector space over $\mathbb{K}$ by the elements $\{e_n \mid n \in \mathbb{Z}\}$ with Lie structure

$$[e_n, e_m] = (m-n)e_{n+m}, \quad n, m \in \mathbb{Z}.$$

Geometric realization:
Over $\mathbb{R}$ it is the Lie algebra of polynomial vector fields $Vect_{pol}(S^1)$ on the circle S^1, $e_n = \exp(\mathrm{i}\, n\, \varphi)\frac{d}{d\varphi}$ with Lie product the usual bracket of vector fields.
This can be complexified, i.e. $\mathbb{K} = \mathbb{C}$: and we obtain the algebra of meromorphic vector fields on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ which are holomorphic outside $\{0\}$ and $\{\infty\}$. In this realization $e_n = z^{n+1}\frac{d}{dz}$.

very important fact: the Witt algebra is a $\mathbb{Z}$-graded Lie algebra by setting $\deg(e_n) := n$

the homogeneous spaces $\mathcal{W}_n$ of degree n are one-dimensional with basis e_n

The eigenspace decomposition of the element e_0, acting via the adjoint action on $\mathcal{W}$ coincides with the decomposition into homogeneous subspaces:

$$[e_0, e_n] = n\, e_n = \deg(e_n)\, e_n.$$

Virasoro algebra $\mathcal{V}$ is the universal one-dimensional central extension of $\mathcal{W}$.
vector space direct sum $\mathcal{V} = \mathbb{K} \oplus \mathcal{W}$
we set for $x \in \mathcal{W}$, $\hat{x} := (0, x)$, and $t := (1, 0)$
basis elements are $\hat{e}_n,\ n \in \mathbb{Z}$ and t with the Lie product

$$
\begin{aligned}
[\hat{e}_n, \hat{e}_m] &= (m - n)\hat{e}_{n+m} - \frac{1}{12}(n^3 - n)\delta_n^{-m}\, t,\\
[\hat{e}_n, t] &= [t, t] = 0,
\end{aligned}
$$

we set $\deg(\hat{e}_n) := \deg(e_n) = n$ and $\deg(t) := 0$ and $\mathcal{V}$ becomes a graded algebra.

we have the following short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{V} \xrightarrow{\ \nu\ } \mathcal{W} \longrightarrow 0\,.$$

the sequence does not split, i.e. it is a non-trivial central extension

the equivalence classes of central extensions are in 1:1 correspondence to the cohomology classes $\mathrm{H}^2(\mathcal{W},\mathbb{K})$

well-known $\dim \mathrm{H}^2(\mathcal{W},\mathbb{K}) = 1$

THE MAIN RESULT

THEOREM

Both the second cohomology of the Witt algebra $\mathcal{W}$ and of the Virasoro algebra $\mathcal{V}$ (over a field $\mathbb{K}$ with char$(\mathbb{K}) = 0$) with values in the adjoint module vanishes, i.e.

$$\mathrm{H}^2(\mathcal{W};\mathcal{W}) = \{0\}, \qquad \mathrm{H}^2(\mathcal{V};\mathcal{V}) = \{0\}.$$

COROLLARY

Both $\mathcal{W}$ and $\mathcal{V}$ are formally and infinitesimally rigid.

Attention: Our cohomology is algebraic cohomology i.e. no restrictions on the cochains are made.

History of this theorem:

- in 1990 Fialowski stated this theorem, but without proof.
- in 2003 Fialowski and myself gave a sketch of a proof, using density arguments and very deep results obtained by Tsujishita, Reshetnikov, and Goncharova. This would be o.k. if we consider continous cohomology. But here we need algebraic cohomology.
- Fortunately, I found a completely elementary proof. arXiv:1111.6625, (in press) Forum Mathematicum.
- Also Fialowski (based on some older calculations done by her) presented an elementary proof, arXiv:1202.3132

Example of a non-trivial family

Lie algebra generated by $V_n, n \in \mathbb{Z}$ over $\mathbb{C}$ with structure

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} & \\ +(e_1-e_2)(e_1-e_3)V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} & \\ +(m-n-2)(e_1-e_2)(e_1-e_3)V_{n+m-4}, & n \text{ odd}, m \text{ even} \end{cases}$$

$(e_3 = -(e_1+e_2))$

Gives a Lie algebra $\mathcal{L}^{(e_1,e_2)}$ for every pair (e_1, e_2).

They are constructed as families of Krichever-Novikov type algebras for the torus (poles at 0 and 1/2 might be allowed).

For $(e_1, e_2) \neq (0,0)$ the algebras $\mathcal{L}^{(e_1,e_2)}$ are not isomorphic to the Witt algebra $\mathcal{W}$, but $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.

(I talked about these families in detail at Bialowieza 2005)

Reduction to degree zero

W be an arbitrary $\mathbb{Z}$-graded Lie algebra, i.e.

$$W = \bigoplus_{n \in \mathbb{Z}} W_n.$$

A k-cochain ψ is homogeneous of degree d if there exists a $d \in \mathbb{Z}$ such that for all $i_1, i_2, \ldots, i_k \in \mathbb{Z}$ and homogeneous elements $x_{i_l} \in W$, of $\deg(x_{i_l}) = i_l$, for $l = 1, \ldots, k$ we have that

$$\psi(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \; \in \; W_n, \quad \text{with} \quad n = \sum_{l=1}^{k} i_l + d$$

we denote the corresponding subspace of degree d homogeneous k-cochains by $C^k_{(d)}(W; W)$

• Every k-cochain can be written as a formal infinite sum

$$\psi = \sum_{d \in \mathbb{Z}} \psi_{(d)},$$

• for a fixed k-tuple of elements only a finite number of the summands will produce values different from zero.

• it is easy to show that the coboundary operators δ_k are operators of degree zero, i.e. applied to a k-cochain of degree d they will produce a $(k+1)$-cochain also of degree d.

• Here only k=2 and k=1 is needed

• ψ is a 2-cocycle if and only if all degree d components $\psi_{(d)}$ will be individually 2-cocycles.

• If $\psi_{(d)}$ is 2-coboundary, i.e. $\psi_{(d)} = \delta_1 \phi$ with a 1-cochain ϕ, then we can find another 1-cochain ϕ' of degree d such that $\psi_{(d)} = \delta_1 \phi'$.

This shows every cohomology class $\alpha \in \mathrm{H}^2(W; W)$ can be decomposed as formal sum

$$\alpha = \sum_{d \in \mathbb{Z}} \alpha_{(d)}, \qquad \alpha_{(d)} \in \mathrm{H}^2_{(d)}(W; W),$$

the latter space consists of classes of cocycles of degree d modulo coboundaries of degree d.

For the rest let W be either $\mathcal{W}$ or $\mathcal{V}$ and assume first $d \neq 0$.

THEOREM

The following hold:

(a) $\mathrm{H}^2_{(d)}(\mathcal{W};\mathcal{W}) = \mathrm{H}^2_{(d)}(\mathcal{V};\mathcal{V}) = \{0\}$, *for* $d \neq 0$.

(b) $\mathrm{H}^2(\mathcal{W};\mathcal{W}) = \mathrm{H}^2_{(0)}(\mathcal{W};\mathcal{W})$, $\quad \mathrm{H}^2(\mathcal{V};\mathcal{V}) = \mathrm{H}^2_{(0)}(\mathcal{V};\mathcal{V})$.

To see this we start with a cocycle of degree $d \neq 0$ and make a cohomological change $\psi' = \psi - \delta_1\phi$ with

$$\phi : W \to W, \quad x \mapsto \phi(x) = \frac{\psi(x, e_0)}{d}.$$

Recall e_0 is the element of either $\mathcal{W}$ or $\mathcal{V}$ which gives the degree decomposition. This implies (note that $\phi(e_0) = 0$)

$$\begin{aligned}\psi'(x, e_0) &= \psi(x, e_0) - \phi([x, e_0]) + [\phi(x), e_0] \\ &= d\,\phi(x) + \deg(x)\phi(x) - (\deg(x) + d)\phi(x) = 0.\end{aligned}$$

Now we evaluate the 2-cocycle condition for the cocycle ψ' on the triple (x, y, e_0) (leave out the cocycle values which vanish due to $\psi'(x, e_0) = 0$)

$$\begin{aligned} 0 =& \psi'([y, e_0], x) + \psi'([e_0, x], y) - [e_0, \psi'(x, y)] \\ &= (\deg(y) + \deg(x) - (\deg(x) + \deg(y) + d))\psi'(x, y) \\ &= -d\psi'(x, y). \end{aligned}$$

As $d \neq 0$ we obtain $\psi'(x, y) = 0$ for all $x, y \in W$.
Hence ψ is a coboundary.

And the theorem is shown.

Witt algebra

• a degree zero cocycle can be written as $\psi(e_i, e_j) = \psi_{i,j} e_{i+j}$

• if it is a coboundary then it is a coboundary of a linear form of degree zero: $\phi(e_i) = \phi_i e_i$

• the systems of $\psi_{i,j}$ and ϕ_i for $i, j \in \mathbb{Z}$ fix ψ and ϕ completely.

• evaluating the cocycle condition for the triple (e_i, e_j, e_k) yields for the coefficients

$$\begin{aligned} 0 =&(j-i)\psi_{i+j,k} - (k-i)\psi_{i+k,j} + (k-j)\psi_{j+k,i} \\ &- (j+k-i)\psi_{j,k} + (i+k-j)\psi_{i,k} - (i+j-k)\psi_{i,j}. \end{aligned} \tag{1}$$

for the coboundary

$$(\delta\phi)_{i,j} = (j-i)(\phi_{i+j} - \phi_j - \phi_i). \tag{2}$$

• hence, ψ is a coboundary if and only if there exists a system of $\phi_k \in \mathbb{K}$, $k \in \mathbb{Z}$ such that

$$\psi_{i,j} = (j-i)(\phi_{i+j} - \phi_j - \phi_i), \quad \forall i, j \in \mathbb{Z}. \tag{3}$$

• A degree zero 1-cochain ϕ will be a 1-cocycle (i.e. $\delta_1\phi = 0$) if and only if

$$\phi_{i+j} - \phi_j - \phi_i = 0.$$

• this has the solution $\phi_i = i\,\phi_1, \forall i \in \mathbb{Z}$

• Hence, given a ϕ we can always find a ϕ' with $(\phi')_1 = 0$ and $\delta_1\phi = \delta_1\phi'$.

• In the following we will always choose such a ϕ' for our 2-coboundaries.

Step 1: cohomological change

start with a 2-cocycle ψ given by the system of $\psi_{i,j}$

modify it by adding a coboundary $\delta_1 \phi$ to obtain $\psi' = \psi - \delta_1 \phi$

Goal is $\psi_{i,1} = 0$ for all $i \in \mathbb{Z}$

Hence, ϕ should fulfill

$$\psi_{i,1} = (1-i)(\phi_{i+1} - \phi_1 - \phi_i) = (1-i)(\phi_{i+1} - \phi_i).$$

(a) Starting from $\phi_0 := -\psi_{0,1}$ we set in descending order for $i \le -1$

$$\phi_i := \phi_{i+1} - \frac{1}{1-i}\,\psi_{i,1}.$$

(b) ϕ_2 cannot be fixed in this way, instead we use

$$\psi_{-1,2} = 3(-\phi_2 - \phi_{-1}), \quad \text{yielding} \quad \phi_2 := -\phi_{-1} - \frac{1}{3}\psi_{-1,2}.$$

Then we have $\psi'_{-1,2} = 0$.

(c) We use again the relation above to calculate recursively in ascending order $\phi_i,\ i \ge 3$

$$\phi_{i+1} := \phi_i + \frac{1}{1-i}\,\psi_{i,1}.$$

For the cohomologous cocycle ψ' we obtain by construction

$$\psi'_{i,1} = 0, \quad \forall i \in \mathbb{Z}, \quad \text{and} \quad \psi'_{-1,2} = \psi'_{2,-1} = 0.$$

Lemma

Let ψ be a 2-cocycle of degree zero such that $\psi_{i,1} = 0, \forall i \in \mathbb{Z}$ and $\psi_{-1,2} = 0$, then ψ will be identical zero.

This says our original cocyle we started with is cohomologically trivial. This shows the main theorem.

It remains to show the lemma.

- The coefficient $\psi_{i,m}$ are called of level m (and of level i by antisymmetry)
- by assumption the cocycle values of level 1 are all zero.
- we will consider $\psi_{i,m}$ for the values of $|m| \le 2$ and finally make ascending and descending induction on m
- specializing the cocycle conditions for the index triple $(i, -1, k)$ gives

$$\begin{aligned} 0 = &-(i+1)\psi_{i-1,k} - (k-i)\psi_{i+k,-1} + (k+1)\psi_{k-1,i} \\ &-(-1+k-i)\psi_{-1,k} + (i+k+1)\psi_{i,k} - (i-1-k)\psi_{i,-1} \end{aligned} \tag{4}$$

and for the triple $(i, 1, k)$

$$0 = (1-i)\psi_{i+1,k} + (k-1)\psi_{k+1,i} + (i+k-1)\psi_{i,k} \tag{5}$$

Using these two relations we can first show that all values of level $m = 0$ and then of level $m = -1$ are zero.

Example: $m = -1$
Setting in (5) $k = -1$ we obtain

$$-(i-1)\,\psi_{i+1,-1} + (i-2)\,\psi_{i,-1} = 0.$$

Hence,

$$\psi_{i,-1} = \frac{i-1}{i-2}\,\psi_{i+1,-1}, \quad \text{for } i \neq 2,$$
$$\psi_{i+1,-1} = \frac{i-2}{i-1}\,\psi_{i,-1}, \quad \text{for } i \neq 1.$$

Starting from $\psi_{1,-1} = -\psi_{-1,1} = 0$ we get (from 1. formula) $\psi_{i,-1} = 0$, for all $i \leq 1$. From the 2. formula we get $\psi_{i,-1} = 0$ for $i \geq 3$ and by assumption $\psi_{2,-1} = \psi_{-1,2} = 0$.

• For $m = -2$ with similar arguments we get $\psi_{i,-2} = 0$ for $i \neq 2, 3$ and the value of $\psi_{2,-2} = -\psi_{3,-2}$ remains undetermined for the moment.

• For $m = 2$ we get $\psi_{i,2} = 0$ for $i \neq -2, -3$ and the value $\psi_{-3,2} = -\psi_{-2,2}$ remains undetermined for the moment.
• Now: consider the index triple $(2, -2, 4)$ in the cocycle condition and we get

$$0 = -2\,\psi_{6,-2} - 8\,\psi_{4,2} + 4\,\psi_{2,-2}.$$

But $\psi_{4,2} = 0$ and $\psi_{6,-2} = 0$ (from $m = 2$ and $m = -2$ discussion). This shows $\psi_{2,-2} = 0$ and all level $m = -2$ and level $m = 2$ values are zero.

• Now the vanishing of all other level m values follow from induction,using (4) and (5).

This proves the lemma and consequently the main theorem for the Witt algebra.

The Virasoro part

We give only a rough sketch

- we start from the short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0\,.$$

- this is also a short exact sequence of Lie modules over $\mathcal{V}$
- we get the part of the long exact cohomology sequence

$$\longrightarrow \mathrm{H}^2(\mathcal{V};\mathbb{K}) \longrightarrow \mathrm{H}^2(\mathcal{V};\mathcal{V}) \longrightarrow \mathrm{H}^2(\mathcal{V};\mathcal{W}) \longrightarrow$$

- we show that naturally $\mathrm{H}^2(\mathcal{V};\mathcal{W}) \cong \mathrm{H}^2(\mathcal{W};\mathcal{W})$ (i.e. every 2-cocycle of $\mathcal{V}$ with values in $\mathcal{W}$ can be changed by a coboundary such that the restriction to $\mathcal{W} \times \mathcal{W}$ defines a 2-cocycle of $\mathcal{W}$)
- we show $\mathrm{H}^2(\mathcal{V};\mathbb{K}) = \{0\}$
- now use $\mathrm{H}^2(\mathcal{W};\mathcal{W}) = 0$ to obtain $\mathrm{H}^2(\mathcal{V};\mathcal{V}) = 0$ $\square$