Einstein Finsler Metrics

and

Ricci flow

Nasrin Sadeghzadeh
University of Qom, Iran

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Outline

- A Survey of Einstein metrics,

- A brief explanation of Ricci flow and its extension to Finsler Geometry,

Ricci flow is used to

- Study the Existence of Einstein Finsler metric of non-constant Ricci scalar.

Einstein Metrics

(Riemann and Finsler)

History

The central assertion of Einstein's general theory of relativity is that

physical space-time is modeled by a 4-manifold that carries a Lorentz metric whose Ricci curvature satisfies the following Einstein field equation.

$$Ric - \frac{1}{2} Sg = T$$

where T is a certain symmetric 2-tensor (the stress-energy tensor)

History

The special case when $T = 0$, the equation reduces to the vacuum Einstein field equation:

$$Ric = \frac{1}{2} Sg$$

It is equivalent to $Ric = 0$, which means that g is a **Einstein metric in mathematical sense of the word**

History

In the development of the theory, Einstein considered adding a term λg to the left side of the equation, where λ is a constant that he called the cosmological constant.

With this modification the vacuum Einstein field equation would be exactly the same as **the mathematicians' Einstein equation.**

Definition of Einstein Riemannian Manifolds

The Einstein equations

$$\boldsymbol{Ric_g = \lambda . g}, \qquad \boldsymbol{\lambda \in R},$$

for a Riemannian metric g are the simplest and most natural set of equations for a metric on a given compact manifold M.

Other Interpretation

If M is a smooth compact n-manifold, $n \geq 3$, then Einstein metrics are critical points of normalized Einstein-Hilbert action functional.

Note that Einstein Hilbert action functional is $\int_M R_g dV_g$,

normalized Einstein Hilbert action functional is $\dfrac{\int_M R_g dV_g}{\left(\int_M dV_g\right)^{\frac{n-2}{n}}}$,

where R_g and dV_g are the scalar curvature and volume form for the Riemannian manifold (M, g).

Relation with other properties (Rie. Schur Lemma)

For manifolds of dimension up to three,
Einstein Riemannian metrics are precisely the same as constant (sectional) curvature metrics

Einstein Finsler metric

A Finsler metric $\boldsymbol{F}$ on an n-dimensional manifold $\boldsymbol{M}$ is called an Einstein metric if there is a scalar function $\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{x})$ on $\boldsymbol{M}$ such that

$$\boldsymbol{Ric} = (\boldsymbol{n} - \mathbf{1})\boldsymbol{K}\boldsymbol{F}^{\mathbf{2}}.$$

Einstein Finsler metrics

Akbar-Zadeh in his paper in titled [1995]

“Generalized Einstein Manifolds” states

Einstein Finsler manifolds are critical points of scalar functional,

the same as Riemannian case. However the integrand function is not the same Riemannian case.

Finsler Ricci Tensor

Akbar-Zadeh introduced the following tensor as **Ricci tensor**

$$\overline{H}_{ij} = \frac{1}{2}\{H(y,y)\}_{.y^i.y^j} = \frac{1}{2}\{H_{pq}y^p y^q\}_{.y^i.y^j},$$

Where $H_{ij} = H_i{}^r{}_{rj}$ is the contracted curvature tensor dependent on Berwald connection.

With assumption $u = \frac{y}{F}$, $H(u,u)$ is called the **direction Ricci curvature.**

Other Interpretation

Einstein metrics as critical point the following functional

$$\boldsymbol{I(g_t) = \int_{SM} \widetilde{H}_t \, d\mu_t,}$$

Where $\widetilde{H}_t = g^{ij}\overline{H}_{ij} - K(x)\overline{H}_t(u,u)$, $K(x)$ is a function on M.

In fact $\boldsymbol{F(g_t)}$ **is a family of Finsler metrics** and $F^0\ (g_t)$ is its sub-family such that in time t the volume of vector bundle depended to metric $g_t \in F^0\ (g_t)$, is equal to 1.

Question?

Does the Einstein Schur lemma hold for arbitrary Finsler metrics?

Is the scalar curvature constant?

Bao and Robles [2003] showed that the following (Einstein Schur Lemma):

The Ricci scalar $Ric(x)$ of any Einstein Randers metric in dimension greater than two is necessarily constant.

To Answer

We use Ricci flow as a tool to investigate the answer of the question.

What is Ricci flow?

Basic Question (Riemannian case)

How can we distinguish the three-dimensional <u>sphere</u> from the other three-dimensional manifolds?

History (Riemannian case)

At the beginning of the 20th century, **Henri Poincaré** was working on the foundations of topology,announced his conjecture

Every simply connected compact 3-manifold (without boundary) is homeomorphic to a 3-sphere.

Poincaré's conjecture became the

base of Ricci flow equation.

History (Riemannian case)

Hamilton's program and Perelman's solution

Hamilton's program was started in his paper in 1982, which he introduced the **Ricci flow** on a manifold and showed how to use it to prove some special cases of the Poincaré conjecture.

History (Riemannian case)

The actual solution wasn't found until Grigori Perelman (of the Steklov institute of Mathematics, Saint petersburg) published his papers using many ideas from Hamilton's work (Ricci flow equation with surjery).

On August 22,2006 ,the ICM awarded Perelman the Fields Medal for his work on the conjecture, but Perelman refused the medal.

Perelman's Proof

He put a Riemannian metric on the unknown simply connected closed 3-manifold .

The idea is to try to improve this metric. The metric is improved using the Ricci flow equations;

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij} ,$$

where g is the metric and R its Ricci curvature, and one hopes that as the time t increases, the manifold becomes easier to understand .

Perelman's theorem

Every closed 3-manifold which admit a metric of positive Ricci curvature also admit a metric of constant positive sectional curvature.

Ricci flow & heat equation

Somewhat like the heat equation

$$\frac{\partial f}{\partial t} = \nabla^2 f$$

except nonlinear.

$$\frac{\partial g_{ij}}{\partial t} = -Ric_{ij}$$

Heat equation evolves a function & Ricci flow evolves a Riemannian metric.

Why Normal Ricci flow equation??

Hamilton found that, sometimes the scalar curvature explodes to $+\infty$ at each point at the same time T and with the same speed.

Then

He showed that it is necessary to form a normalization that makes the volume constant.

What is Normal Ricci flow equation

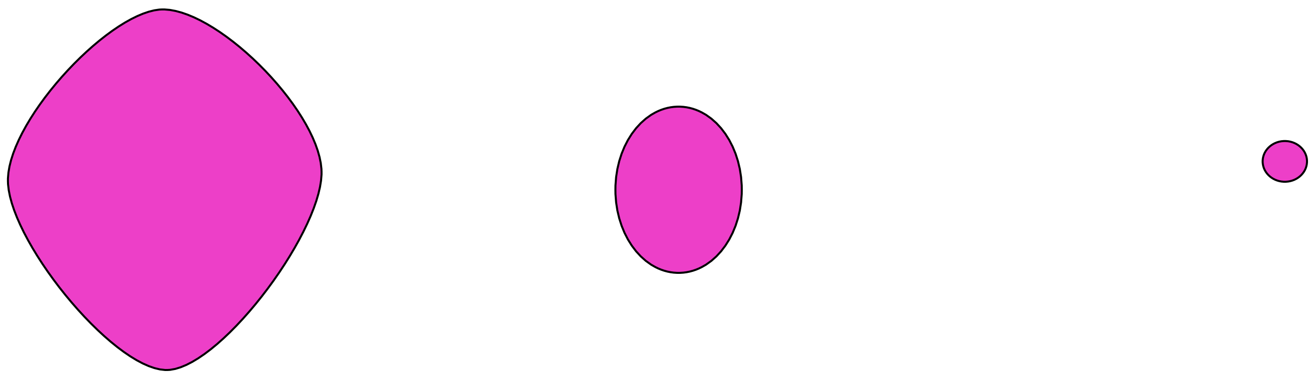
If M is compact we can normalize the equation by requiring that the flow keeps the volume of M constant. i.e.

$$Vol(g_{ij}) = \int_M dV = 1$$

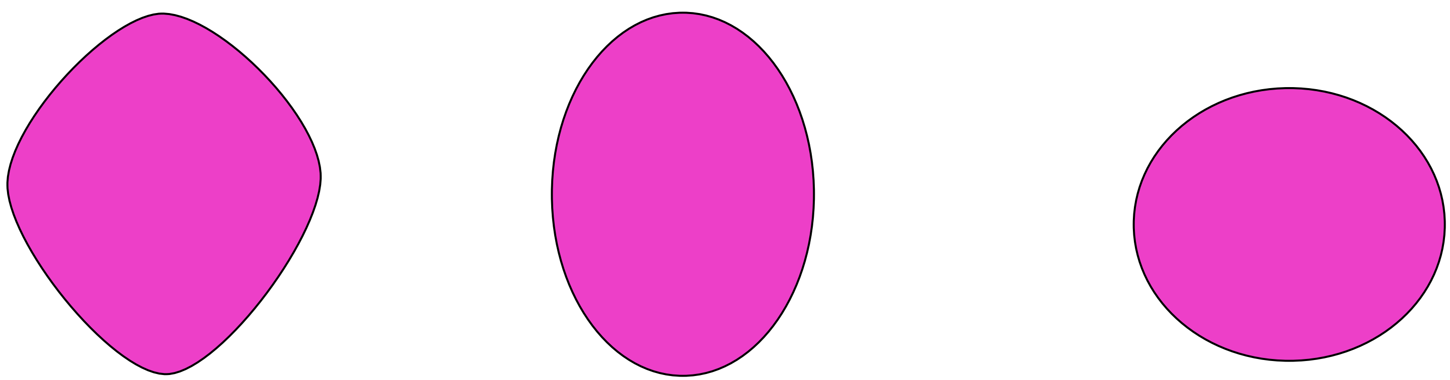
Therefore the normal Ricci flow equation is found as follows

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij} + \frac{2}{n}\int_M \rho dV \, .$$

Unnormalized Ricci flow



Normalized Ricci flow



Ricci flow
in
Finsler geometry

Chern question

Does every manifold admit an Einstein Finsler metric or a Finsler metric of constant flag curvature?

It is hoped that the Ricci flow in Finsler geometry eventually proves to be viable for addressing Chern's question.

Why is there Ricci flow equation in Finsler space?

In principle, the same equation can be used in the Finsler setting,

Because both g_{ij} and Ric_{ij} have been generalized to that broader framework, albeit gaining a y dependence in the process.

Un-normalized equation

Bao [2007] have stated a scalar equation instead of this tensor evolution equation.

He contracted the equation with l^i, l^j and via Euler's theorem is gotten

$$\partial_t \log F = -Ric, \quad F(t=0) = F_0$$

Normalized equation

If M is compact, then so is SM, and we can normalize the above equation by requiring that the flow

keeps the volume of SM constant.

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Normalized Equation

To normalize the proposed Ricci flow, we replace its equation by

$$\partial_t \log F = -Ric + C(t),$$

and determine $C(t)$ so that the volume of SM remains constant under the evolution of F and then

$$C(t) = \frac{1}{Vol_{SM}} \int_{SM} Ric\, dV_{SM} = Avg(Ric).$$

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Tensor Ricci flow equation

Finsler Ricci flow equation in the tensor form is the same as Riemannian case.

It can be used the Akbar-Zadeh's version of Ricci tensors as

$$Ric_{ij} = (R_k{}^m{}_{ml} y^k y^l)_{.y^i .y^j}.$$

Einstein Metric of Non-Constant Ricci Scalar

Fixed Point of Ricci Flow Equation

A Finsler metric $\boldsymbol{F}$ is a fixed point of the un-normal Ricci flow if and only if it is Ricci flat.

A compact Finsler manifold $(\boldsymbol{M}, \boldsymbol{F})$ is a fixed point of the normalized Finsler Ricci flow if and only if it is an Einstein Finsler metric of constant Ricci scalar.

Finsler self-similar Solution

A Finsler tensor field $(M, g(t))$ on smooth manifold M such that $F(t)$ is a solution of the scalar un-normalized Finsler Ricci flow on a time t ∈ $(-\varepsilon, \varepsilon)$, is called

Finsler self-similar Solution of Ricci flow

if there exist a function $\sigma(t, x)$ and diffeomorphism φ_t of M such that

$$g(t) \; = \sigma(t,x)\varphi_t{}^{*} \; (g0).$$

Finsler Ricci Solitons

A Finsler tensor field (M, g_0) on smooth manifold M such that F_0 is a fixed Finsler manifold and the identity

$$2Ric_0 = L_X g_0 + 2\mu(x) g_0,$$

holds for some functions $\mu(x)$ and some complete vector field X on M, is called a Finsler Ricci Soliton.

Equivalency of these two definitions

If $(\boldsymbol{M}, \boldsymbol{g}(\boldsymbol{t}))$ is a solution of the un-normal Ricci flow as a self similar solitons,

then there exist a vector field $\boldsymbol{X}$ on

$\boldsymbol{M}$ such that $(\boldsymbol{M}, \boldsymbol{g_0}, \boldsymbol{X})$ solve

$$-2Ric0 = L_X g_0 + 2\mu(x) g_0,$$

And

Equivalency of these two definition

Conversely,

For any solution $(\boldsymbol{M}, \boldsymbol{g_0}, \boldsymbol{X})$ of

$$-2Ric0 = L_X g_0 + 2\mu(x) g_0,$$

there exist a 1-parameter families of functions $\boldsymbol{\sigma}(\boldsymbol{t}, \boldsymbol{x})$ and diffeomorphisms $\boldsymbol{\varphi_t}$of $\boldsymbol{M}$ such that $(\boldsymbol{M}, \boldsymbol{g}(\boldsymbol{t}))$ becomes a solution of the un-normal Ricci flow when $\boldsymbol{g}(\boldsymbol{t})$ is defined by

$$g(t) = \sigma(t, x)\varphi_t{}^* (g0).$$

Einstein Finsler metrics of non-constant Ricci Scalar

Let

- F_0 be a projrctively flat Finsler metric on M,

- $F_t = h(t, x)F_0$,
where $h := h(t, x)$ is a positive continues function on M.

Then

F_t is Ricci constant iff $(\frac{h'}{h})_{;x^l} = 0$.

Relation between Geodesic coefficients F_t and F_0

Let

$G_t{}^i$ geodesic coefficients of F_t , $G_0{}^i$ geodesic coefficients of F_0

$$G^i{}_t = G^i{}_0 + Py^i + Q^i = \widetilde{H}^i + (P_0 + 2P)y^i,$$

where

$$P = \frac{h_{;x^k}y^k}{2h}, \qquad P_0 = \frac{F_{0;x^k}y^k}{2F_0},$$

$$\widetilde{H}^i = -F_0{}^2 g_0{}^{il} P_{.l} := u_0{}^{il} P_{.l}$$

Relation between Ricci Scalar F_t and F_0

$$R_t = \frac{R_0}{h^2} - {\alpha_0}^l \frac{P_{.l}}{h^2} - \beta^{jl}\frac{P_{.j;x^l}}{h^2} - 2\gamma^{jl}\frac{P_{.j}P_{.l}}{h^2},$$

Where

$${\alpha_0}^l = -\left(\frac{1}{{F_0}^2}\right)\{2(n-1)(2P_0 y^l + {u_0}^{ml} P_{0.m}) + 2{u_0}^{ml}{}_{;x^m} - y^j {u_0}^{ml}{}_{;x^j.m}\},$$

$$\beta^{jl} = -\left(\frac{1}{{F_0}^2}\right)\{2{u_0}^{jl} + 2{u_0}^{ml}{}_{.m} y^j - 2(n-1)y^j y^l\},$$

$$\gamma^{jl} = -\left(\frac{1}{{F_0}^2}\right)\{4(n-1)({u_0}^{jl} + y^j y^l) + 2{u_0}^{jl}\, {u_0}^{ml}{}_{.j.m} - {u_0}^{jk}{}_{.m}{u_0}^{ml}{}_{.j}.$$

Final Equation (PDE)

Since F_t satisfying Ricci flow equation one gets

$$(h^2)' = -2R_0 + \alpha_0{}^l \, \frac{h_{;x^l}}{h} - \beta^{jl}\left(\frac{h_{;x^l}}{h}\right)_{;x^j} - 2\gamma^{jl}\,\frac{h_{;x^j}\, h_{;x^l}}{h},$$

Therefore

In order to find an Einstein Finsler metric $F_t = h(t,x)F_0$ with **non-constant Ricci scalar** it suffices to have $h(t,x)$ that satisfying the above equation.

Some References

- Akbar-Zadeh H. , Generalized Einstein manifolds, J. Geom. Phys.17(1995), 342-380.
- Bao D. , On two curvature-driven problems in Riemann-Finsler geometry, Advanced Studies in Pure Mathematics XX, 2007.
- Cao H. -D. and Zhu X. -P. , Hamilton–Perelman's proof of the Poincar'e conjecture and the geometrization conjecture, Asian J. Math. 10 (2006), 165–495; arXiv: math. DG/ 0612069.

Some References

- Hamilton R. S. , Four Manifolds with Positive Curvature Operator, J. Diff. Geom. 24, 153-179, 1986.
- SadeghZadeh N. and Razavi A. , Ricci Flow equation on C-reducible metrics, International Journal of Geometric Methods in Modern Physics, (2011), DOI No: 10.1142/S0219887811005385.
- VACARU S. , ON GENERAL SOLUTIONS OF EINSTEIN EQUATIONS, International Journal of Geometric Methods in Modern PhysicsVol. 8, No. 1 (2011) 9–21c.

Thank you