Poisson sigma models as constrained Hamiltonian systems

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Outline

Poisson manifold

Differentiable manifold $(M, \{,\})$ with Poisson bracket

- $\{,\} : C^\infty(M) \times C^\infty(M) \mapsto C^\infty(M)$
- bilinear, antisymmetric $\{f,g\} = -\{g,f\}$
- Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$
- Leibnitz rule $\{fg,h\} = f\{g,h\} + \{f,h\}g$

Poisson bivector Π

$$\Pi = \Pi^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \in \Lambda^2 TM \qquad \Pi^{ij} \in C^\infty(M)$$

$$\{f,g\} = \Pi(\mathrm{d}f, \mathrm{d}g) = \Pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \qquad \Pi^{ij}(x) = \{x^i, x^j\}$$

$$\Pi^{ij} = -\Pi^{ji} \qquad \Pi^{ij} \frac{\Pi^{kl}}{\partial x^j} + \Pi^{kj} \frac{\Pi^{li}}{\partial x^j} + \Pi^{lj} \frac{\Pi^{ik}}{\partial x^j} = 0$$

Casimir functions

f is called Casimir function if $\{f,g\} = 0 \quad \forall g \in C^\infty(M)$

Poisson σ-model [Schaller & Strobl, 1994],[Calvo & Falceto, 2006]

Building blocks

- Σ - differentialble manifold, $\mathtt{dim}\Sigma = 2$, coordinates σ^μ
- (M, Π) - Poisson manifold $\mathtt{dim}M = n$, coordinates x^i

Dynamical fields

- vector bundle map $(X, \tilde{A}) : T\Sigma \mapsto T^*M$
  base manifold map $X : \Sigma \mapsto M$, in coordinates $X^i = x^i \circ X(\sigma)$
  total spaces map $\tilde{A} : T\Sigma \mapsto T^*M$
- or $A : T\Sigma \mapsto \Gamma(\Sigma, X^*(T^*M))$ $\quad \langle A, V\rangle(\sigma) := \tilde{A}(V_\sigma) \in T^*_{X(\sigma)}M$
  $A(\sigma) = A_i(\sigma)\mathtt{d}x^i \mid_{X(\sigma)}$

σ-model action and field equations

Action $\quad S_{X,A} = \int_\Sigma A_i \wedge \mathtt{d}X^i + \frac{1}{2}\Pi^{ij}(X)A_i \wedge A_j \quad$ where $dX^i = X^*(dx^i)$

Equations of Motion $\quad \mathtt{d}X^i + \Pi^{ij}(X)A_j = 0 \qquad \mathtt{d}A_i + \frac{1}{2}\frac{\partial \Pi^{jk}}{\partial x^i}(X)A_j \wedge A_k = 0$

First order string action

Action for a bosonic string

$$S_{str} = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{\mathrm{det}(h)} \left(G_{ij}(X) h^{\mu\nu} + \epsilon^{\mu\nu} B_{ij}(X)\right) \partial_\mu X^i \partial_\nu X^j$$

where h is a metric on Σ, G is a metric on M, B is a 2-form on M.

Seiberg-Witten limit [Baulieu *et al.*, 2002]

$$S^{fo}_{str} = \int_\Sigma A_i \wedge \mathrm{d}X^i + \pi\alpha' E^{ij}(X) A_i \wedge \star A_j + \frac{1}{2}\Pi^{ij}(X) A_i \wedge A_j$$

using Equations of Motion for A_i $\qquad \mathrm{d}X^i + \Pi^{ij}(X)A_j + E^{ij}(X) \star A_j = 0$

classical equivalence if

$$(G+B)^{-1} = E + \frac{1}{2\pi\alpha'}\Pi$$

Seiberg-Witten limit $\alpha' \to 0$ while (G, B) fixed, we get the action of PSM.

Gauge theory for $SU(2)$

Yang-Mills action

- $\mathfrak{su}(2)$-valued 1-form $A = A_i T^i$
- curvature 2-form $F = (\mathrm{d}A_i + \frac{1}{2} c_i^{jk} A_j \wedge A_k) T^i$, where $c_i^{jk} = \epsilon^{ijk}$

$$S_{YM} = -\frac{1}{4g} \int_\Sigma Tr(F \wedge \star F)$$

Action for a linear PSM with a Casimir function

- $\mathbb{R}^3$ with Poisson bivector $\Pi^{ij} = \sum_{k=1}^3 \epsilon^{ijk} x^k$
- Casimir function $R^2 = \sum_{k=1}^3 x^k x^k$ on M, volume form ω on Σ

$$S_{YM}^{fo} = \int_\Sigma X^i (\mathrm{d}A_i + \frac{1}{2} c_i^{jk} A_j \wedge A_k) - \frac{g}{2} \omega \sum_{k=1}^{3} X^k X^k$$

Equations of Motion $X^i = \frac{1}{g} \star F_i$ give classical equivalence with S_{YM}.

For $g \to 0$ describes action of a PSM.

Local symmetry [Ikeda, 1994], [Schaller & Strobl, 1994]

Local symmetry of the action of PSM

$$\epsilon_j = \epsilon_j(\sigma) \qquad \delta_\epsilon X^i = \epsilon_j \Pi^{ji}(X) \qquad \delta_\epsilon A_i = \mathrm{d}\epsilon_i + \frac{\partial \Pi^{jk}}{\partial x^i}(X) A_j \epsilon_k$$

generalization of Y-M $SU(2)$ symmetry - linear Π

$$S_{X+\delta X, A+\delta A} - S_{X,A} = \int_\Sigma \mathrm{d}(\mathrm{d}X^i \epsilon_i)$$

For Π fulfilling Jacobi identity and $\partial\Sigma = \emptyset$ or $\epsilon_{|\partial\Sigma} = 0$

Algebra of these transformations is closed only on-shell

$$\begin{aligned}
[\delta_\epsilon, \delta_{\epsilon'}] X^i &= \epsilon_j \epsilon'_k \frac{\partial \Pi^{jk}}{\partial x^l} \Pi^{li} \\
[\delta_\epsilon, \delta_{\epsilon'}] A_i &= \mathrm{d}(\epsilon_j \epsilon'_k \frac{\partial \Pi^{jk}}{\partial x^i}) + \frac{\partial \Pi^{jk}}{\partial x^i} A_j (\epsilon_o \epsilon'_p \frac{\partial \Pi^{op}}{\partial x^k}) - \epsilon_j \epsilon'_k \frac{\partial^2 \Pi^{jk}}{\partial x^i \partial x^l} (\mathrm{d}X^l + \Pi^{lm} A_m)
\end{aligned}$$

Constrained system formalism

With $A_i(\tau,\sigma) = A_{i\tau}d\tau + A_{i\sigma}d\sigma$, action gives

$$S_{X,A} = -\int_{\tau_1}^{\tau_2} d\tau \int_\sigma d\sigma \left(A_{i\sigma}X^i_{,\tau} - A_{i\tau}(X^i_{,\sigma} + \Pi^{ij}A_{j\sigma}) \right)$$

Hamiltonian formalism

coordinates $X^i(\tau,\sigma)$, momenta $P_i(\tau,\sigma) = A_{i\sigma}(\tau,\sigma)$ and canonical Hamiltonian

$$\begin{aligned} H &= \int_\sigma d\sigma \left(P_i(\sigma)X^i_{,\tau}(\sigma) - \mathcal{L}(\sigma) \right) \\ &= \int_\sigma d\sigma A_{i\tau}(\sigma)\left(X^i_{,\sigma}(\sigma) + \Pi^{ij}(X(\sigma))A_{j\sigma}(\sigma) \right) = \int_\sigma d\sigma \lambda_i(\sigma)\Phi^i(\sigma) \end{aligned}$$

Equations of Motion - constrained system, $H_0 = 0$

$$\begin{aligned} \dot{X}^i(\sigma) &= \frac{\delta H_0}{\delta P_i(\sigma)} + \frac{\delta}{\delta P_i(\sigma)} \int d\sigma' \lambda_j(\sigma')\Phi^j(\sigma'), \\ \dot{P}_i(\sigma) &= -\frac{\delta H_0}{\delta X^i(\sigma)} - \frac{\delta}{\delta X^i(\sigma)} \int d\sigma' \lambda_j(\sigma')\Phi^j(\sigma'), \\ \Phi^i(\sigma) &= 0. \end{aligned}$$

Constraints and gauge transformations

Constraints $\Phi^i(\sigma) = X'^i(\sigma) + \Pi^{ik}(X(\sigma))P_k(\sigma)$ are first class

$$\{\Phi^i(\sigma'), \Phi^j(\sigma'')\} = -\delta(\sigma' - \sigma'')\Pi^{ij}_{,n}(X(\sigma'))\Phi^n(\sigma') \approx 0$$
$$\{\Phi^i(\sigma), H\} = \lambda_j(\sigma)\Pi^{ji}_{,n}(X(\sigma))\Phi^n(\sigma) \approx 0$$

Gauge transformations

$$\delta_\epsilon X^i(\sigma) = \int d\sigma' \epsilon_j(\sigma')\{X^i(\sigma), \Phi^j(\sigma')\} = \epsilon_j(\sigma)\Pi^{ji}(X(\sigma))$$
$$\delta_\epsilon P_i(\sigma) = \int d\sigma' \epsilon_j(\sigma')\{P_i(\sigma), \Phi^j(\sigma')\} = \epsilon'_i(\sigma) + P_k(\sigma)\Pi^{kj}_{,i}(X(\sigma))\epsilon_j(\sigma)$$
$$\delta_\epsilon \lambda_i(\sigma) = \dot{\epsilon}_i(\sigma) + \lambda_k(\sigma)\Pi^{kj}_{,i}(X(\sigma))\epsilon_j(\sigma)$$

Algebra of gauge transformations

$$[\delta_\epsilon, \delta_{\epsilon'}]X^i = \epsilon_j\epsilon'_k \frac{\partial \Pi^{jk}}{\partial x^l}\Pi^{li} = \delta_{\epsilon''}X^i$$
$$[\delta_\epsilon, \delta_{\epsilon'}]A_i = \delta_{\epsilon''}A_i - \epsilon_j\epsilon'_k \frac{\partial^2 \Pi^{jk}}{\partial x^i \partial x^l}(\mathrm{d}X^l + \Pi^{lm}A_m)$$

Poisson-Lie group

Poisson map

$(M, \{,\}), (N, \{,\})$ Poisson manifolds, $F : N \mapsto M$ is Poisson map if $(\forall f, g \in C^\infty(M))(\{f, g\}_M \circ F = \{f \circ F, g \circ F\}_N)$

Multiplicativity of Π

compatibility of group and Poisson structure - $\mu : G \times G \mapsto G$ is Poisson map with $\{f, g\}_{M \times N}(x, y) = \{f(., y), g(., y)\}_M(x) + \{f(x, .), g(x, .)\}_N(y)$ Poisson structure on $G \times G$

$$\Pi(gg') = (L_{\star g} \otimes L_{\star g})\Pi(g') + (R_{\star g'} \otimes R_{\star g'})\Pi(g)$$

Relation to bialgebra structures

for (G, Π) connected and simply connected P-L group with algebra $\mathfrak{g}$ exists unique bialgebra structure $(\mathfrak{g}, \delta)$ such that $\delta = D\Pi$ and vice versa.

Construction of Π on the group G

Manin triple $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$

$\mathfrak{d} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$, $\mathfrak{g}, \tilde{\mathfrak{g}}$ are subalgebras of $\mathfrak{d}$, isotropic w.r.t. non-degenerate symmetric ad-invariant scalar product $\langle , \rangle_{\mathfrak{d}}$

$$[T_a, T_b] = c_{ab}^c T_c \qquad [\tilde{T}^a, \tilde{T}^b] = \tilde{c}_c^{ab} \tilde{T}^c \qquad [T_a, \tilde{T}^b] = c_{ca}^b \tilde{T}^c + \tilde{c}_a^{bc} T_c$$

`adjoint representation of` G `on` $\mathfrak{d}$ $\quad Ad_{g^{-1}} = \begin{pmatrix} a(g)^T & b(g)^T \\ 0 & d(g)^T \end{pmatrix}$

Poisson-Lie structure on G

in the basis of right-invariant fields

$$\Pi(g) = -b(g).a^{-1}(g)$$

Sklyanin bracket gives the same result

G connected Lie group with $\mathfrak{g}$, δ given by an r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$, then

$$\Pi(g) = (L_{\star g} \otimes L_{\star g})(r) - (R_{\star g} \otimes R_{\star g})(r)$$

Poisson-Lie T-Duality

σ-model

- $\Phi : \Sigma \mapsto G, \Phi \in C^\infty$
- $\Sigma \subset \mathbb{R}^2$ with Minkowski metric
- d-dimensional Lie-group G with $\mathcal{F} = \mathcal{G} + \mathcal{B}$
- action

$$S_{\mathcal{F}}(\Phi) = \int_\Sigma d\sigma_+ d\sigma_- \partial_- \Phi^\mu \mathcal{F}_{\mu\nu}(\Phi) \partial_+ \Phi^\nu$$

Dualizable σ-models

generalized symmetry condition [Klimčík & Ševera, 1995]

$$(\mathcal{L}_{V_a}\mathcal{F})_{\mu\nu} = \mathcal{F}_{\mu\kappa} v_b^\kappa \widetilde{f}_a^{bc} v_c^\lambda \mathcal{F}_{\lambda\nu}$$

self-consistency leads to the Drinfeld double - $D \equiv (G|\widetilde{G})$ is a Lie group whose Lie algebra $\mathfrak{d}$ admits a decomposition $\mathfrak{d} = \mathfrak{g} \oplus \tilde{\mathfrak{g}}$ into a pair of subalgebras, maximally isotropic with respect to a symmetric, ad-invariant, nondegenerate bilinear form $\langle\, .\, ,\, .\, \rangle_{\mathfrak{d}}$

Construction of dualizable σ-models

σ-model on G

given orthogonal subspaces $\mathcal{E}^{\pm} \in \mathfrak{d}$ - graph of E_0, dual decomposition $l = g\tilde{h} = \tilde{g}h$ in

$$\langle \partial_{\pm} l l^{-1}, \mathcal{E}^{\pm} \rangle = 0$$

$$E(g) = (a(g) + E_0 \cdot b(g))^{-1} \cdot E_0 \cdot d(g)$$

in right-invariant fields setting $E(g) = \left(E_0^{-1} + \Pi(g)\right)^{-1}$ with $\Pi(g) = b(g)a(g)^{-1}$

$$\mathcal{F}(g) = e \cdot E(g) \cdot e^T \qquad (\partial_{\pm} g g^{-1})^a = \partial_{\pm} \phi^{\mu} e^a_{\mu}(g)$$

$\mathcal{F}(g)$ solves the generalized symmetry condition and we have σ-model with Equations of Motion

$$\partial_+(\partial_- \tilde{h}\tilde{h}^{-1})_c - \partial_-(\partial_+ \tilde{h}\tilde{h}^{-1})_c + \widetilde{f}_c^{ab}(\partial_- \tilde{h}\tilde{h}^{-1})_a(\partial_+ \tilde{h}\tilde{h}^{-1})_b = 0.$$

$$(\partial_+ \tilde{h}\tilde{h}^{-1})_a + E_{ab}(g)(g^{-1}\partial_+ g)^b = 0$$

$$(\partial_- \tilde{h}\tilde{h}^{-1})_a - (g^{-1}\partial_- g)^b E_{ba}(g) = 0$$

Dual σ-model

Dual σ-model on $\tilde{G}$

model on G

$$\mathcal{E}^+ = Span(T^i + E_0^{ij}\tilde{T}_j) \quad \rightarrow \quad g^{-1}\mathcal{E}^+ g = Span(T^i + E^{ij}(g)\tilde{T}_j)$$

model on $\tilde{G}$ for invertible E_0

$$\mathcal{E}^+ = Span(\tilde{T}_j + (E_0^{-1})_{ji}T^i) \quad \rightarrow \quad \tilde{g}^{-1}\mathcal{E}^+\tilde{g} = Span(\tilde{T}_j + \tilde{E}(\tilde{g})_{ji}T^i)$$

model on $\tilde{G}$ for non-invertible E_0

$$\mathcal{E}^+ = Span(T^i + E_0^{ij}\tilde{T}_j) \quad \rightarrow \quad \tilde{g}^{-1}\mathcal{E}^+\tilde{g} = Span(T^i + \tilde{E}(\tilde{g})^{ij}\tilde{T}_j)$$
$$\tilde{E}(\tilde{g}) = d(\tilde{g})^{-1}(E_0 \cdot a(\tilde{g}) + b(\tilde{g}))$$

Equations of Motion and Lagrangian

$$\tilde{L} = -\lambda_{+i}\tilde{E}^{ij}(\tilde{g})\lambda_{-j} + \lambda_{+i}(\tilde{g}^{-1}\partial_-\tilde{g})^i + \lambda_{-i}(\tilde{g}^{-1}\partial_+\tilde{g})^i$$

for $\lambda_{\pm i} = (\partial_\pm hh^{-1})_i$ null vectors of $\tilde{E}$ act as Lagrange multipliers

$$\partial_+\lambda_{-c} - \partial_-\lambda_{+c} - f_c^{ab}\lambda_{+a}\lambda_{-b} = 0$$
$$(\tilde{g}^{-1}\partial_-\tilde{g})^a + \tilde{E}^{ab}(\tilde{g})\lambda_{-b} = 0$$
$$(\tilde{g}^{-1}\partial_+\tilde{g})^a - \lambda_{+b}\tilde{E}^{ba}(\tilde{g}) = 0$$

Example - R^2 gravity

$\mathfrak{su}(2)$-Yang-Mills is formulated on the double $(1 \mid 9)$, what about others?

4 and 6 dimensional Drinfeld doubles classified [Šnobl & Hlavatý, 2002]

R^2 gravity [Schaller & Strobl, 1994]

$$\mathcal{L}_{R^2} = \frac{1}{4}\int_M d^2x\sqrt{|\ det\ g\ |}(\frac{1}{4}R^2 + 1)$$

in Einstein-Cartan variables (e^1, e^2) zweibein, ω connection 1-form

$$\mathcal{L}_{R^2} = \int_M X(de^1 - \omega \wedge e^2) + Y(de^2 + \omega \wedge e^1) + Zd\omega + (\frac{1}{4} - Z^2)e^1 \wedge e^2$$

for $X^i = (X, Y, Z), A_i = (e^1, e^2, \omega)$ we have PSM with

$$\{X, Y\} = -Z^2 + \frac{1}{4} \qquad \{Y, Z\} = X \qquad \{Z, X\} = Y$$

induces Lie algebra structure of Bianchi9 ($\mathfrak{so}(3)$) so it can be built on $(1 \mid 9)$ as

$$\{X, Y\} = \frac{1}{2} - z \qquad \{Y, Z\} = X \qquad \{Z, X\} = Y$$

or Bianchi8 ($\mathfrak{sl}(2, \mathbb{R})$), however using $(1 \mid 7_0), (2i \mid 7_0), (2ii \mid 7_0)$ etc. needs simpler coordinate transformation

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Thank you for your attention.