GINIBRE ENSEMBLES and INTEGRABLE HIERARCHIES

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REAL GINIBRE RANDOM MATRICES

Ginibre (1965):
On the space of real $N \times N$ matrices define the probability measure

$$d\mu(X) = e^{-\mathrm{tr}(XX^{\dagger})} \prod_{i,j} dX_{ij} \qquad \left(= \prod_{i,j} e^{-X_{ij}^2} dX_{ij} \quad \text{real case} \right)$$

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Partition functions for, respectively, the real Ginibre ensemble, real Ginibre-Girko ensemble and induced Ginibre-Girko ensembles are

$$\int d\mu(X)\,, \quad \int e^{-\nu \mathrm{tr} X^2} d\mu(X)\,, \quad \int \det\left(XX^\dagger\right)^L e^{-\nu \mathrm{tr} X^2} d\mu(X)$$

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X may be real (real GE), complex (complex GE), quaternion real (qu-r GE)

On the desk

To write down real symmetric measure $\prod_{i\geq j} dR_{ij} e^{-R_{ij}^2}$ and compare to Ginibre real.

To explain what is the quaternion and quaternion real matrix and also quaternion self-dual one (the last is Hermitian, the first generally not)

On applications

Wigner and Dyson had physical reasons to consider orthogonal, unitary and symplectic ensembles. Physical systems are described by Hamiltonian operators which are Hermitian. Ginibre had no physical reasons and introduced non-Hermitian analogues from "mathematical"point of view, having a hope that they may be of use in future. Indeed at present Ginibre ensembles are in a focus of attention in many fields (mainly in quantum chaos problems):

APPLICATIONS OF REAL, COMPLEX AND QUATERNION REAL GINIBRE ENSEMBLES

APPLICATIONS: Ginibre ensembles appeared in the studies of

- quantum chromodynamics
- dissipative quantum maps
- scattering in chaotic quantum systems
- growth processes
- fractional quantum-Hall effect
- Coulomb plasma
- stability of complex biological and neural networks
- directed quantum chaos in randomly pinned superconducting vortices
- delayed time series in financial markets
- random operations in quantum information theory

On the desk: Deformation of Wigner-Dyson ensembles. The importance

In 2D quantum gravity Kazakov and Brezin considered a deformation of the Gauss measure of WD unitary ensemble by cubic term in the exponent.
Then it was natural to add the whole Taylor series (and we did it).
Then it turned out that the partition function of the matrix model is a tau function of TL hierarchy deformation parameters being higher times.
Higher times were related to combinatorial problems of fat graphs counting.
Later WD orhtogonal ensemble was deformed by Adler-van Moerbeke and related to the so-called Pfaff lattice. Then J. van de Leur related it to BKP of Kac-van de Leur which is correct integrable hierarchy. Also it was related to Mobius graphs counting by Mulase. He showed the duality between orthogonal and symplectic ensembles.
Importance of deformation in rich links to many other problems.

Theorem Both for real and quaternion real Ginibre ensembles the partition function

$$\tau_N(L,\mathbf{t},\mathbf{s}) = \int \det X^L \, e^{\sum_{m=1}^{\infty}\left(t_m \mathrm{tr} X^m - s_m \mathrm{tr} X^{-m}\right)} \, d\mu(X)$$

is a tau function for the "large" BKP hierarchy introduced by V.Kac and J. van de Leur [2] separately with respect to the set N, L, t where $\mathbf{t} = (t_1, t_2, \dots)$ and to the set N, L, s where $\mathbf{s} = (s_1, s_2, \dots)$.

V. Kac and J. van de Leur, "The Geometry of Spinors and the Multicomponent BKP and DKP Hierarchies", CRM Proceedings and Lecture Notes **14** (1998) 159-202

A. Yu. Orlov, T.Shiota and K. Takasaki, "Pfaffian structures and certain solutions to BKP hierarchies I. Sums over partitions", arXiv: math-ph/12014518

Mehta, Khoruzhenko, Sommers:

After applying the Schur decomposition $X = U(\Lambda + \Delta)U^{\dagger}$, where U is orthogonal, Λ is block-diagonal and Δ has nonzero blocks only above Λ

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they arrive at

$$d\mu(X) = e^{-2\mathrm{Tr}(\Delta\Delta^{\dagger}+\Lambda\Lambda^{\dagger})} D\Delta D\Lambda \Big| \prod{}' \left(U^{-1}dU\right)_{ij} (\lambda_i - \lambda_j) \Big|$$

with the dashed product running over non-zero entries in the lower triangle of $U^{\dagger}dU$.

Joint Probability Distribution of Eigenvalues in Ginibre Ensembles

Complex case

$$d\mu(X) = \prod_{i<j} |z_i - z_j|^2 \prod_{i=1}^{N} e^{-|z_i|^2} d^2 z_i \times (*)$$

where $(*)$ is independent of eigenvalues: similar to normal matrices

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Then it is known that the deformation is governed by the two-component 2D TL (O.Zaboronsky; J.Harnad, A.O.)

Joint Probability Distribution of Eigenvalues

Quaternion real case

$$d\mu(X) = \prod_{i<j} |z_i - z_j|^2 |z_i - \bar{z}_j|^2 \prod_{i=1}^{N} |z_i - \bar{z}_i| e^{-|z_i|^2} d^2 z_i \times (*)$$

where $(*)$ is independent of eigenvalues.

For X real: there are real eigenvalues λ_i and complex conjugated pairs $(z_i, \bar{z}_i)$

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Real case

$$d\mu(X) = \sum_{k=0} |\Delta_k(\lambda, z, \bar{z})| \prod_{i=1}^{N-2k} e^{-\lambda_i^2} d\lambda_i \prod_{i=1}^{N} \mathrm{erfc}(\sqrt{2}|\Im z_i|) e^{-\Re z_i^2} d^2 z_i \times (*)$$

where $(*)$ is independent of eigenvalues and where

$$\Delta_k(\lambda, z, \bar{z}) = \prod_{i<j\leq N-2k} (\lambda_i - \lambda_j) \prod_{i,j} |\lambda_i - z_j|^2 \prod_{i<j\leq k} |z_i - z_j|^2 |z_i - \bar{z}_j|^2$$

Quaternionic real Ginibre ensemble with the deformed measure

$$\int \det\left(XX^\dagger\right)^L e^{\sum_{m=1}\left(t_m \mathrm{tr} X^m - s_m \mathrm{tr} X^{-m}\right)} e^{-\mathrm{tr} XX^\dagger} \prod_{i,j} dX_{ij}$$

Fermionic representation for quaternion real GE

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is the following 2-BKP [] tau function

$$= \langle N+L|\,\Gamma(\mathbf{t})\, e^{\frac{1}{2}\int_{\mathbb{C}}(z-\bar z)\psi(z)\psi(\bar z)e^{-|z|^2} d^2 z}\, \bar\Gamma(\mathbf{s})\,|L\rangle$$

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For 2-BKP tau function see

A. Yu. Orlov, T.Shiota and K. Takasaki, "Pfaffian structures and certain solutions to BKP hierarchies I. Sums over partitions", arXiv: math-ph/12014518

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where

$$\Phi_c = \int_{\mathbb{C}_+} \mathrm{erfc}\left(\frac{|z-\bar{z}|}{\sqrt{2}}\right) \psi(z)\psi(\bar{z}) d^2 z$$

$$\Phi_r = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{sign}(x_1 - x_2)\psi(x_1)\psi(x_2) dx_1\, dx_2 + \int_{\mathbb{R}} \psi(x)\phi_0\, dx$$

Fermionic representation for complex GE

Complex Ginibre ensemble with the deformed measure

$$\int \det(XX^\dagger)^L e^{\sum_{m=1}\left(t_m \mathrm{tr} X^m + t'_m \mathrm{tr}(X^\dagger)^m - s_m \mathrm{tr} X^{-m} - s'_m \mathrm{tr}(X^\dagger)^{-m}\right) - \mathrm{tr} XX^\dagger} \prod_{i,j} dX_{ij}$$

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is the following two-component 2D Toda lattice tau function

$$= \langle L+N, L-N|\, \Gamma(\mathbf{t},\mathbf{t}')\, e^{\int_{\mathbb{C}} \psi^{(1)}(z)\psi^{\dagger(2)}(\bar{z}) e^{-|z|^2} d^2 z}\, \bar{\Gamma}(\mathbf{s},\mathbf{s}')\, |L, L\rangle$$

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This tau function is the same that used in the model of normal matrices

J. Harnad and A.Yu. Orlov, "Fermionic construction of partition functions for two-matrix models and perturbative Schur function expansions", *J. Phys.* **A** 8783-8810 (2006)

Outproduct

A fermionic representation in a natural way allows

- Pfaffian representation for partition functions and correlators (Wick theorem)
- Schur function expansion of integrals ($\int \cdots \int \to \sum \cdots \sum$) via rewriting Fermi fields via Fermi modes
- calculation of characteristic polynomials via Baker functions (Wick theorem)
- constraints and string equations ('Ward identities')

bibliography

1.

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