Duality for multiple vector bundles

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1. Introduction

Recall the duality between Lie algebras and linear Poisson spaces:

> *The dual $\mathfrak{g}^*$ of a (finite-dim) Lie algebra $\mathfrak{g}$ has a linear Poisson structure and if a (finite-dim) vector space V has a linear Poisson structure, then V^* has a Lie algebra structure; these processes are inverse.*

Applying the same process to the bracket of sections of the tangent bundle TM of a manifold, we see that it is dual in the same way to the symplectic structure on T^*M.

These two dualities are instances of the duality for Lie algebroids:

> *The dual A^* of a Lie algebroid A has a linear Poisson structure and if a vector bundle E has a linear Poisson structure, then E^* has a Lie algebroid structure; these processes are inverse.*

The work in this talk grew out of considering multiple versions of this duality. These arise for several reasons; I mention just one:

For any Poisson manifold P the cotangent bundle T^*P has a Lie algebroid structure.

So given a Lie algebroid A there is a Lie algebroid structure on T^*A^*. This is a vector bundle over A^* but there is also a vector bundle structure over A, due to the canonical diffeomorphism $T^*A^* \to T^*A$ (valid for any vector bundle). With these two structures T^*A^* is a *double vector bundle*.

2. TA

Before returning to T^*A, consider TA for A a vector bundle on M.

TA is a vector bundle on A (of course), but there is a second vector bundle structure on TA, this one with base TM.

The projection is $T(q)$ where $q\colon A \to M$ is the projection of A.

The zero section is $T(0)$, the addition is $T(+)$, ...everything works, because T preserves diagrams. (BTW, to emphasize this process, I write $T(f)$ instead of df for any map of manifolds.)

We show these two structures in the diagram

$$\begin{tikzcd} TA \arrow[r, "T(q)"] \arrow[d, "p_A"'] & TM \arrow[d, "p_M"] \\ A \arrow[r, "q"] & M \end{tikzcd}$$

This is a double vector bundle (definition shortly). The diagram is not meant to be read as a morphism; it should be read as a mathematical structure in its own right.

3. Exact sequences

There are two short exact sequences associated with TA.

Tq is a map of vector bundles so has a kernel.

$$A \times_M A \longrightarrow TA \xrightarrow{Tq} TM$$

A vector $\xi \in TA$ which is annulled by Tq is vertical, so is tangent to a fibre, so consists of a base-point in some fibre, and a vector in that fibre.

The kernel is the inverse image bundle of A over itself.

p_A is also a map of vector bundles and has a kernel.

$$A \times_M TM \longrightarrow TA \xrightarrow{p_A} A\ .$$

A vector $\xi \in TA$ which is annulled by p_A is on the zero section. Given $\xi \in T_{0_m}A$, project ξ to $X = T(q)(\xi) \in TM$. Then $\xi - T(0)(X)$ is vertical, so identifies with an $e \in A_m$.

The kernel is the inverse image of A over $TM \to M$.

A *connection in* A can be defined as a map $A \times_M TM \to TA$ which is right-inverse to each of Tq and p_M (and bilinear).

4. Structures on TA

The two structures on TA are compatible in the sense that the maps defining each structure are linear with respect to the other.

For the additions this means that given four elements, $\xi_i \in TA$, $i = 1, \ldots, 4$,

$$\begin{tikzcd} \xi_i \arrow[r] \arrow[d] & X_i \arrow[d] \\ a_i \arrow[r] & m \end{tikzcd} \quad \text{of} \quad \begin{tikzcd} TA \arrow[r, "T(q)"] \arrow[d, "p_A"'] & TM \arrow[d, "p_M"] \\ A \arrow[r, "q"] & M \end{tikzcd}$$

Then

$$(\xi_1 + \xi_2) \underset{TM}{+} (\xi_3 + \xi_4) = (\xi_1 \underset{TM}{+} \xi_3) + (\xi_2 \underset{TM}{+} \xi_4).$$

Here $+$ is the standard addition of tangent vectors and $\underset{TM}{+}$ is the addition in $TA \to TM$.

This is the *interchange law*. It is the main defining condition for a *double vector bundle*.

5. Double vector bundles

A *double vector bundle* is a manifold D with two vector bundle structures, over bases A and B, each of which is a vector bundle on a manifold M, such that the structure maps of $D \to A$ (the bundle projection q_A, the addition $\underset{A}{+}$, the scalar multiplication, the zero section) are morphisms of vector bundles with respect to the other structure.

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

The condition that the addition $\underset{A}{+}$ is a morphism with respect to the other structure is the interchange law

$$(d_1 \underset{A}{+} d_2) \underset{B}{+} (d_3 \underset{A}{+} d_4) = (d_1 \underset{B}{+} d_3) \underset{A}{+} (d_2 \underset{B}{+} d_4).$$

6. Comments

Double vector bundles go back to the 1950s (Dombrowski) and were used in the 1960s and 1970s in some accounts of connection theory (Dieudonné, Besse) and theoretical mechanics (Tulczyjew). The first systematic account was given by Pradines (1977).

They are not the same as 2-vector bundles. I'll say something about this at the end, but everything for double (and multiple) vector bundles is for finite-dimensional smooth manifolds and all algebraic structures are strict.

7. 'Decomposed' example

There will be more examples shortly. For now, a very simple example.

Take vector bundles A and B on base M. The manifold $A \times_M B$ can be given two vector bundle structures.

First, regard $A \times_M B$ as $q_A^! B$, the pullback of B over q_A. Next, regard $A \times_M B$ as $q_B^! A$, the pullback of A over q_B.

This is a double vector bundle (a very simple one).

Now consider three vector bundles A, B, C on the same base M, and the manifold $A \times_M B \times_M C$.

First, form the Whitney sum bundle $B \oplus C \to M$ and take the pullback over q_A. This gives a vector bundle $q_A^!(B \oplus C)$ over base A.

Next, form the Whitney sum bundle $A \oplus C \to M$ and take the pullback over q_B. This gives a vector bundle $q_B^!(A \oplus C)$ over base B.

With these two structures, $D := A \times_M B \times_M C$ is a double vector bundle, called *decomposed*.

Every double vector bundle is isomorphic to a decomposed double vector bundle (not usually in a natural way).

Note: The Whitney sum $A \oplus B \oplus C$ is a vector bundle over M. This is a special feature of decomposed bundles.

8. Duality

$D \to A$ is a vector bundle so can be dualized as usual. There is no a priori reason to expect that the result will form a double vector bundle.

$$
\begin{tikzcd}
D \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
D \overset{\times}{\vert} A \arrow[r] \arrow[d] & ? \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
$$

However . . .

Write C for the set of all elements of D which project to zero in both structures.

These are closed under addition, and the two additions coincide, due to the interchange law.
So C is a vector bundle over M.
C is the *core* of D.

$$
\begin{tikzcd}
c \arrow[r] \arrow[d] & 0^B_m \arrow[d] \\
0^A_m \arrow[r] & m
\end{tikzcd}
$$

9. Short exact sequences

The bundle projection $D \to B$ is a morphism of vector bundles over $A \to M$. Write K_{hor} for its kernel. Every element of K_{hor} is the sum (uniquely) of a core element and a zero element in $D \to A$.

$$\begin{tikzcd} k \arrow[r] \arrow[d] & 0^B_m \arrow[d] \\ a \arrow[r] & m \end{tikzcd} \quad \text{equals} \quad \begin{tikzcd} c \arrow[r] \arrow[d] & 0^B_m \arrow[d] \\ 0^A_m \arrow[r] & m \end{tikzcd} \quad \text{plus (over } B\text{)} \quad \begin{tikzcd} \widetilde{0}_a \arrow[r] \arrow[d] & 0^B_m \arrow[d] \\ a \arrow[r] & m \end{tikzcd}$$

where $c = k -_B \widetilde{0}_a$.

The addition in K_{hor} turns out to correspond to adding the core elements. So K_{hor} is the inverse image bundle $q_A^! C$ and we have a short exact sequence

$$0 \longrightarrow q_A^! C \longrightarrow D \longrightarrow q_A^! B \longrightarrow 0$$

(Shriek denotes inverse image.)

10. *Short exact sequences, p2*

The dual of the short exact sequence

$$0 \longrightarrow q_A^! C \longrightarrow D \longrightarrow q_A^! B \longrightarrow 0$$

is

$$0 \longrightarrow q_A^! B^* \longrightarrow D \mathbin{\overset{\times}{\uparrow}} A \longrightarrow q_A^! C^* \longrightarrow 0$$

This suggests that there may be a double vector bundle

$$\begin{tikzcd} D \mathbin{\overset{\times}{\uparrow}} A \arrow[r] \arrow[d] & C^* \arrow[d] \\ A \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} D \mathbin{\overset{\times}{\uparrow}} B \arrow[r] \arrow[d] & B \arrow[d] \\ C^* \arrow[r] & M \end{tikzcd}$$

and this is so. Likewise there is a double vector bundle $D \mathbin{\overset{\times}{\uparrow}} B$.

Note: The windmill symbol $\overset{\times}{\uparrow}$ denotes *the ordinary vector bundle dual*. I use this distinctive symbol because after several iterations the usual symbol $*$ becomes confusing.

11. Example: duals of TA

For $D = TA$ the core is A. Consider: the kernel of $TA \to A$ is the vectors along the zero section. And the kernel of $TA \to TM$ is the vertical vectors. Vertical vectors are tangent to the fibres and at zero can be identified with points of the fibres.

$$\begin{tikzcd} TA \arrow[r] \arrow[d] & TM \arrow[d] \\ A \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} T^*A \arrow[r] \arrow[d] & A^* \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

What is the dual of TA over TM ? Apply the tangent functor to $A \times_M A^* \to \mathbb{R}$ and we get $TA \times_{TM} T(A^*) \to \mathbb{R}$, also a non-degenerate pairing. So

$$\begin{tikzcd} TA \arrow[r] \arrow[d] & TM \arrow[d] \\ A \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} T(A^*) \arrow[r] \arrow[d] & TM \arrow[d] \\ A^* \arrow[r] & M \end{tikzcd}$$

12. The duals are dual

Theorem: $D \overset{\times}{\uparrow} A \to C^*$ and $D \overset{\times}{\uparrow} B \to C^*$ are themselves dual.

'PROOF': Take $\Phi \in D \overset{\times}{\uparrow} A$ and $\Psi \in D \overset{\times}{\uparrow} B$ projecting to same $\kappa \in C^*$. Say $\Phi \mapsto a \in A$ and $\Psi \mapsto b \in B$.

$$\begin{tikzcd} \Phi \arrow[r] \arrow[d] & \kappa \arrow[d] \\ a \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} \Psi \arrow[r] \arrow[d] & b \arrow[d] \\ \kappa \arrow[r] & M \end{tikzcd} \qquad \begin{tikzcd} d \arrow[r] \arrow[d] & b \arrow[d] \\ a \arrow[r] & M \end{tikzcd}$$

Take any $d \in D$ which projects to a and b. The pairing is

$$\langle \Phi, \Psi \rangle_{C^*} = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B.$$

The subtraction ensures that the RHS is well-defined.

These are duals as double vector bundles.

Note: We could define

$$\langle \Phi, \Psi \rangle_{C^*} = -\langle \Phi, d \rangle_A + \langle \Psi, d \rangle_B.$$

Apart from the choice of signs, the pairing is unique.

13. The duality group

Now write X for dualization in the vertical structure and Y for dualization in the horizontal.

$$
\begin{tikzcd}
D \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
D^X \arrow[r] \arrow[d] & C^* \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
D^{XY} \arrow[r] \arrow[d] & C^* \arrow[d] \\
B \arrow[r] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
D^{XYX} \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & M
\end{tikzcd}
$$

The final double vector bundle is the ‘flip’ of the first. There is no canonical sense in which the two can be identified.

Now interchange X and Y :

$$
\begin{tikzcd}
D \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
D^Y \arrow[r] \arrow[d] & B \arrow[d] \\
C^* \arrow[r] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
D^{YX} \arrow[r] \arrow[d] & A \arrow[d] \\
C^* \arrow[r] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
D^{YXY} \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & M
\end{tikzcd}
$$

The results *are* canonically isomorphic. Briefly, $XYX = YXY$.

Together with $X^2 = Y^2 = I$ this shows that X, Y generate the symmetric group of order 6. Write $\mathscr{DF}_2$ for this group.

In effect $\mathscr{DF}_2$ is the symmetric group on $\{A, B, C^*\}$.

14. *Triple case*

Before going on to the triple case, it's reasonable to ask: Why go further ?

$$\text{Lie algebroid on } A \Longrightarrow \text{Poisson structure on } A^* \Longrightarrow \text{Lie algebroid on } T^*(A^*)$$

(Double vector bundle)

In a similar way the cotangent of a double vector bundle is a triple vector bundle. Any study of bracket structures on a double vector bundle will lead to working with triples.

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
TD \arrow[rr] \arrow[dr] \arrow[dd] & & TB \arrow[dr] \arrow[dd] & \\
& TA \arrow[rr] \arrow[dd] & & TM \arrow[dd] \\
D \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& A \arrow[rr] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
T^*D \arrow[rr] \arrow[dr] \arrow[dd] & & D \overset{\times}{\uparrow} B \arrow[dr] \arrow[dd] & \\
& D \overset{\times}{\uparrow} A \arrow[rr] \arrow[dd] & & C^* \arrow[dd] \\
D \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& A \arrow[rr] & & M
\end{tikzcd}$$

And there is always curiosity. As it turns out the answer in the triple case is surprising.

15. Triple vector bundles

From here on I am describing joint work with Alfonso Gracia-Saz (LMP, 2009).

$$\begin{tikzcd}
E_{1,2,3} \arrow[rr, "X"] \arrow[dr] \arrow[dd] & & E_{2,3} \arrow[dr] \arrow[dd] & \\
& E_{1,3} \arrow[rr] \arrow[dd] & & E_3 \arrow[dd] \\
E_{1,2} \arrow[rr] \arrow[dr] & & E_2 \arrow[dr] & \\
& E_1 \arrow[rr] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{1,2,3} \overset{\times}{\uparrow} E_{2,3} \arrow[rr] \arrow[dr] \arrow[dd] & & E_{2,3} \arrow[dr] \arrow[dd] & \\
& E_{3,12} \overset{\times}{\uparrow} E_3 \arrow[rr] \arrow[dd] & & E_3 \arrow[dd] \\
E_{2,31} \overset{\times}{\uparrow} E_2 \arrow[rr] \arrow[dr] & & E_2 \arrow[dr] & \\
& E^*_{123} \arrow[rr] & & M
\end{tikzcd}$$

On the RHS is E^X. Imagine calculating E^{XYXZ} this way . . . it gets unwieldy very quickly.

Each face of E is a double vector bundle and has a core (denoted $E_{1,23}$, $E_{2,31}$, $E_{3,12}$).

Further, there is the set of all elements $e \in E_{1,2,3}$ which project to zeros in all three of $E_{1,2}$, $E_{2,3}$ and $E_{3,1}$. It is a vector bundle on base M, denoted E_{123} and called the *ultracore*.

For brevity write $E_0 = E^*_{123}$.

16. Duals of triple vector bundles

$$
\begin{tikzcd}
E_{1,2,3} \arrow[rr, "X"] \arrow[dd] \arrow[dr] & & E_{2,3} \arrow[dd] \arrow[dr] & \\
& E_{1,3} \arrow[rr] \arrow[dd] & & E_3 \arrow[dd] \\
E_{1,2} \arrow[rr] \arrow[dr] & & E_2 \arrow[dr] & \\
& E_1 \arrow[rr] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
E_{1,2,3} \overset{\times}{\uparrow} E_{2,3} \arrow[rr] \arrow[dd] \arrow[dr] & & E_{2,3} \arrow[dd] \arrow[dr] & \\
& E_{3,12} \overset{\times}{\uparrow} E_3 \arrow[rr] \arrow[dd] & & E_3 \arrow[dd] \\
E_{2,31} \overset{\times}{\uparrow} E_2 \arrow[rr] \arrow[dr] & & E_2 \arrow[dr] & \\
& E_0 \arrow[rr] & & M
\end{tikzcd}
$$

X leaves E_2 and E_3 fixed and interchanges E_1 with E_0.

	E_1	E_2	E_3	E_0
X	E_0	E_2	E_3	E_1
Y	E_1	E_0	E_3	E_2
Z	E_1	E_2	E_0	E_3

So the group of dualization functors acts as S_4 on E_1, E_2, E_3 and E_0.

17. Duality for triple vector bundles

We have a surjection $\mathscr{DF}_3 \to S_4$ and want the kernel.

S_4 is generated by $\sigma_1 = (01)$, $\sigma_2 = (02)$, $\sigma_3 = (03)$. These are subject to

$$\sigma_i^2 = 1, \qquad (\sigma_i\sigma_j)^3 = 1, \qquad (\sigma_i\sigma_j\sigma_i\sigma_k)^2 = 1,$$

for i, j, k distinct.

We know that $X^2 = Y^2 = Z^2 = 1$, and that $(XY)^3 = \cdots = 1$.

Is it also true that $(XYXZ)^2 = 1$?

To settle this, look at the 'automorphisms' of E.

18. Statomorphisms

First the double vector bundle case.

A *statomorphism* $\varphi\colon\ D \to D$ is an automorphism which induces the identity on A, B and the core C.

We may as well consider just the decomposed case, $D = A \times_M B \times_M C$.

Then φ is

$$\varphi(a, b, c) = (a, b, c + \xi(a, b))$$

where $\xi\colon\ A \times_M B \to C$ is a bilinear map. We usually write $\xi\colon\ A \otimes B \to C$.

In the triple case we have the cores E_{12}, E_{13}, E_{23} of the lower faces and the ultracore E_{123}. So an element of a decomposed triple vector bundle is $(e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123})$,

and a statomorphism is determined by six bilinear maps

$$\begin{gathered}(1,2,03)\colon\ E_1 \otimes E_2 \to E_{12}, \quad (1,3,02)\colon\ E_1 \otimes E_3 \to E_{13}, \quad (2,3,01)\colon\ E_2 \otimes E_3 \to E_{23}, \\ (1,23,0)\colon\ E_1 \otimes E_{23} \to E_{123}, \quad (2,13,0)\colon\ E_2 \otimes E_{13} \to E_{123}, \quad (3,12,0)\colon\ E_3 \otimes E_{12} \to E_{123},\end{gathered}$$

and one trilinear map: $(1,2,3,0)\colon\ E_1 \otimes E_2 \otimes E_3 \to E_{123}$.

19. Statomorphisms, p2

Now a dualization operator will act on statomorphisms. In the double case applying X to $\xi\colon\ A \otimes B \to C$ sends it to $-\xi\colon\ A \otimes C^* \to B^*$. (We use the same letter for $A \otimes B \to C$ and the rearrangements $A \otimes C^* \to B^*$, $A \otimes B \otimes C^*$, ...) The minus sign comes from the minus sign in the pairing of duals.

The triple case :

	$(1,2,03)$	$(1,3,02)$	$(2,3,01)$	$(1,23,0)$	$(2,13,0)$	$(3,12,0)$
X	$-(2,13,0)$	$-(3,12,0)$	$(2,3,01)$	$-(1,23,0)$	$-(1,2,03)$	$-(1,3,02)$
Y	$-(1,23,0)$	$(1,3,02)$	$-(3,12,0)$	$-(1,2,03)$	$-(2,13,0)$	$-(2,3,01)$
Z	$(1,2,03)$	$-(1,23,0)$	$-(2,13,0)$	$-(1,3,02)$	$-(2,3,01)$	$-(3,12,0)$

We can now calculate the effect of a word such as $(XYXZ)^2$ on the statomorphisms and we get

	$(1,2,03)$	$(1,3,02)$	$(2,3,01)$	$(1,23,0)$	$(2,13,0)$	$(3,12,0)$
X	$-(2,13,0)$	$-(3,12,0)$	$(2,3,01)$	$-(1,23,0)$	$-(1,2,03)$	$-(1,3,02)$
YX	$(2,13,0)$	$(2,3,01)$	$-(3,12,0)$	$(1,2,03)$	$(1,23,0)$	$-(1,3,02)$
XYX	$-(1,2,03)$	$(2,3,01)$	$(1,3,02)$	$-(2,13,0)$	$-(1,23,0)$	$(3,12,0)$
$XYXZ$	$-(1,2,03)$	$(2,13,0)$	$(1,23,0)$	$-(2,3,01)$	$-(1,3,02)$	$-(3,12,0)$
$(XYXZ)^2$	$(1,2,03)$	$-(1,3,02)$	$-(2,3,01)$	$-(1,23,0)$	$-(2,13,0)$	$(3,12,0)$

20. Statomorphisms, p3

So $(XYXZ)^2$ does not act as the identity on the statomorphisms. This certainly suggests that $(XYXZ)^2$ is a nonidentity element of the kernel. However, we have not yet made clear what the group $\mathscr{DF}_3$ is and when an element *is* the identity.

Duality of ordinary vector bundles is a contravariant functor. For triple vector bundles, X, Y, Z are contravariant functors (on suitable categories) and XY, for example, is a covariant functor.

Defn: Two words W_1 and W_2 in X, Y, Z define the same element of $\mathscr{DF}_3$ if they induce the same permutation on E_1, E_2, E_3, E_0 and if $W_1 W_2^{-1}$ is naturally isomorphic to the identity through statomorphisms.

Consider a word W in X, Y, Z. If W is in the kernel, then it is a covariant (auto)functor on the category of triple vector bundles,

Theorem: The action of W on the set of statomorphisms is the identity if and only if W is naturally isomorphic to the identity functor through statomorphisms.

So $(XYXZ)^2 \neq 1$. Equivalently, $(XYX)Z \neq Z(XYX)$. So ‘flipping’ in the XY-plane does not commute with dualizing in the Z direction.

21. Structure of the group $\mathscr{DF}_3$

Write K_4 for the kernel of $\mathscr{DF}_3 \to S_4$.

We have that $(XYXZ)^2 \neq 1$ is in K_4. Likewise $(YZYX)^2$ and $(ZXZY)^2$ are in K_4, and (with 1) form the Klein 4-group K_4.

So $\mathscr{DF}_3$ is an extension of S_4 by the Klein four–group.

$$1 \to K_4 \to \mathscr{DF}_3 \to S_4 \to 1.$$

In particular $\mathscr{DF}_3$ has order 96.

Action of S_4 on K_4: for (01) use X:

$$X(XYXZ)^2X = X(XYXZ)(XYXZ)X = YXZXYXZX = (YXZX)^2 = (YZXZ)^2 = (ZXZY)^2,$$

and so on. The extension is not semi-direct.

- Question: What do the (non-identity) elements in the kernel represent? They have order 2 so are like classical duality operations (but are covariant).
  However they affect only the "internal structure". They are "covert."
- The main consequence of the determination of $\mathscr{DF}_3$ may be expressed as:

  > In addition to the identity $(XY)^3 = 1$ (and its conjugates), which arise from the duality of the duals of a double vector bundle, in the triple case there is only one further identity, namely $(XYXZ)^4 = 1$.

22. Four-fold case

$\mathscr{DF}_4$ is an extension of S_5 by $K_5 = C_2 \times C_2 \times C_2 \times C_2 \times C_2$.

$$1 \to K_5 \to \mathscr{DF}_4 \to S_5 \to 1.$$

$\mathscr{DF}_4$ has order $3,840$.

$\mathscr{DF}_4$ has nontrivial centre C_2.

Defining relations:

For ordinary duality we have $X^2 = I$.

For the duality of doubles we have, as well, $(XY)^3 = I$.

For the duality of triples we have, as well, $(XYXZ)^4 = I$.

For $n = 4$ we have, as well, two unexpected words of length 24 and 32.

23. Remarks

- These groups seem unreasonably large. However neither S_{n+1} nor K_{n+1} is interesting in itself; it is the extension that is significant, especially the set of relations needed to define the group (using as generators the basic dualizations).

- This work arose from studying bracket structures on double vector bundles.

  If a double vector bundle D has Lie algebroid structures on each side, then they are compatible if and only if structures obtained from dualization, namely $D \mathbin{\overset{\times}{\uparrow}} A \to C^*$ and $D \mathbin{\overset{\times}{\uparrow}} B \to C^*$, form a Lie bialgebroid over C^*.

  Th. Voronov has formulated this as a commutativity condition for homological vector fields (that is, for two Q-manifold structures on D).

- The groups $\mathscr{D}\mathscr{F}_n$ are not invariants of any one n-fold vector bundle, but rather of the whole class of n-fold vector bundles. As far as we know, these groups have not been studied before.

24. References

See

A. Gracia-Saz and K. Mackenzie, *"Duality functors for triple vector bundles," Lett. Math. Phys.* **90**, 2009, 175 – 200.

The dual of a double object (VB-groupoid) is due to

J. Pradines, *"Remarque sur le groupoïde cotangent de Weinstein-Dazord," C. R. Acad. Sci. Paris Sér. I Math.* **306**, 1988, 557–560.

That the duals of a double vector bundle are in duality, comes from:

K. Mackenzie, *"On symplectic double groupoids and the duality of Poisson groupoids," Int. J. Math.* **10**, 1999, 435 – 456.

More detail on the double case is in Chapters 3 and 9 of:

K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids,* London Mathematical Society Lecture Note Series, no. 213, CUP, 2005.

25. References, p2

For duality of doubles used to define double Lie algebroids, see

K. Mackenzie, *Ehresmann doubles and Drinfel'd doubles for Lie algebroids and Lie bialgebroids,* J. Reine Angew. Math., **658**, *193–245,* 2011.

For the formulation in terms of commuting homological vector fields, see

Th. Voronov, *Q-manifolds and Mackenzie theory,* Comm. Math. Phys., *to appear.*

For the cases $n \geqslant 4$,

A. Gracia-Saz and K. Mackenzie, *"Duality functors for n-fold vector bundles,"* to arrive on arXiv shortly.