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QUANTUM DEFORMATIONS OF D=4 SUPERPOINCARÉ ALGEBRAS AND THEIR EUCLIDEAN COUNTERPARTS

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Based on: arXiv: 1112.1936v2 [hep-th], JHEP 06(2012)154 (A. Borowiec, J.L., M. Mozrzymas, V.N. Tolstoy)

1. INTRODUCTION

a) ***Non-SUSY case.***

Need for quantum space-time symmetries → to describe covariantly theories with noncommutative (NC) space-time

$$[x_\mu, x_\nu] = 0 \Rightarrow [\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa^2}\theta_{\mu\nu}(\kappa\hat{x})$$

where

$$\theta_{\mu\nu}(\kappa\hat{x}) = \theta^{(0)}_{\mu\nu} + \kappa\,\theta^{(1)\rho}_{\mu\nu}\hat{x}_\rho + \kappa^2\,\theta^{(2)\rho\tau}_{\mu\nu}\hat{x}_\rho\hat{x}_\tau + \ldots$$

DFR or canonical	Lie-algebraic deformation	quadratic deformation

By the presence of constant tensors $\theta^{(0)}_{\mu\nu}, \theta^{(1)\rho}_{\mu\nu} \ldots$ the classical Poincaré symmetry is broken

noncommutative space-time → breaking of classical relativistic invariance

However one can find Hopf-algebraic quantum deformation of Poincaré symmetries $\mathcal{P}^{3;1} = O(3,1) \ltimes T_4$ which keeps the non-commutativity relation for $\hat{x}_\mu$ the same in all deformed Poincaré frames

$$\hat{g} \triangleright \left([\hat{x}_\mu, \hat{x}_\nu] - \frac{1}{\kappa^2}\theta_{\mu\nu}(\kappa\hat{x})\right) = 0$$

where $\hat{g}$ is the generator of deformed Poincaré-Hopf algebra H

$\hat{g} \triangleright \ldots$ is the action of generator $\hat{g}$ on Hopf(algebra) module $\mathbb{X} = (\mathcal{M}(\hat{x}), \cdot)$, determined by the coproduct

$$\Delta(\hat{g}) = \hat{g}_{(1)} \otimes \hat{g}_{(2)}$$

Action $\triangleright$ given by Hopf-algebraic formula:

$$\hat{g} \triangleright (x \cdot y) = (g_{(1)} \triangleright x)(g_{(2)} \triangleright y) \qquad x, y \in \mathbb{X}$$

Remark: If the module $\mathbb{X}$ is noncommutative necessarily

$$\Delta(\hat{g}) \neq \Delta^T(\hat{g}) = \hat{g}_{(2)} \otimes \hat{g}_{(1)} \quad \leftarrow \quad \text{nonsymmetric coproduct}$$

Important link in NC geometry:

noncommutative space-time $\longleftrightarrow$ covariance under quantum relativistic symmetries

Remark: The quantum covariance condition selects only some models of NC space-time

Examples:

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa^2}\theta^{(0)}_{\mu\nu} \longleftrightarrow$$ quantum Poincaré algebra obtained by canonical twist $\exp(\frac{i}{2}\theta^{\mu\nu}P_\mu \wedge P_\nu)$

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa}\hat{x}_i \qquad [\hat{x}_i, \hat{x}_j] = 0 \longleftrightarrow$$ κ-deformation of Poincaré algebra algebra - can not be obtained by twist

Advantage of twist deformation - explicit formula for the star multiplication $\star$ representing products of $f(\hat{x}) \subset \mathcal{M}(f;\cdot)$

$$f(\hat{x}) \cdot f(\hat{y}) \xrightarrow{\text{Weyl map}} f(x) \star f(y) = (\bar{F}_{(1)} \triangleright f(x))(\bar{F}_{(2)} \triangleright f(y))$$

$F = F_{(1)} \otimes F_{(2)}$ - twist $\Longrightarrow$ $F^{-1} = \bar{F}_{(1)} \otimes \bar{F}_{(2)}$ (inverse twist)

b) ***SUSY case.***

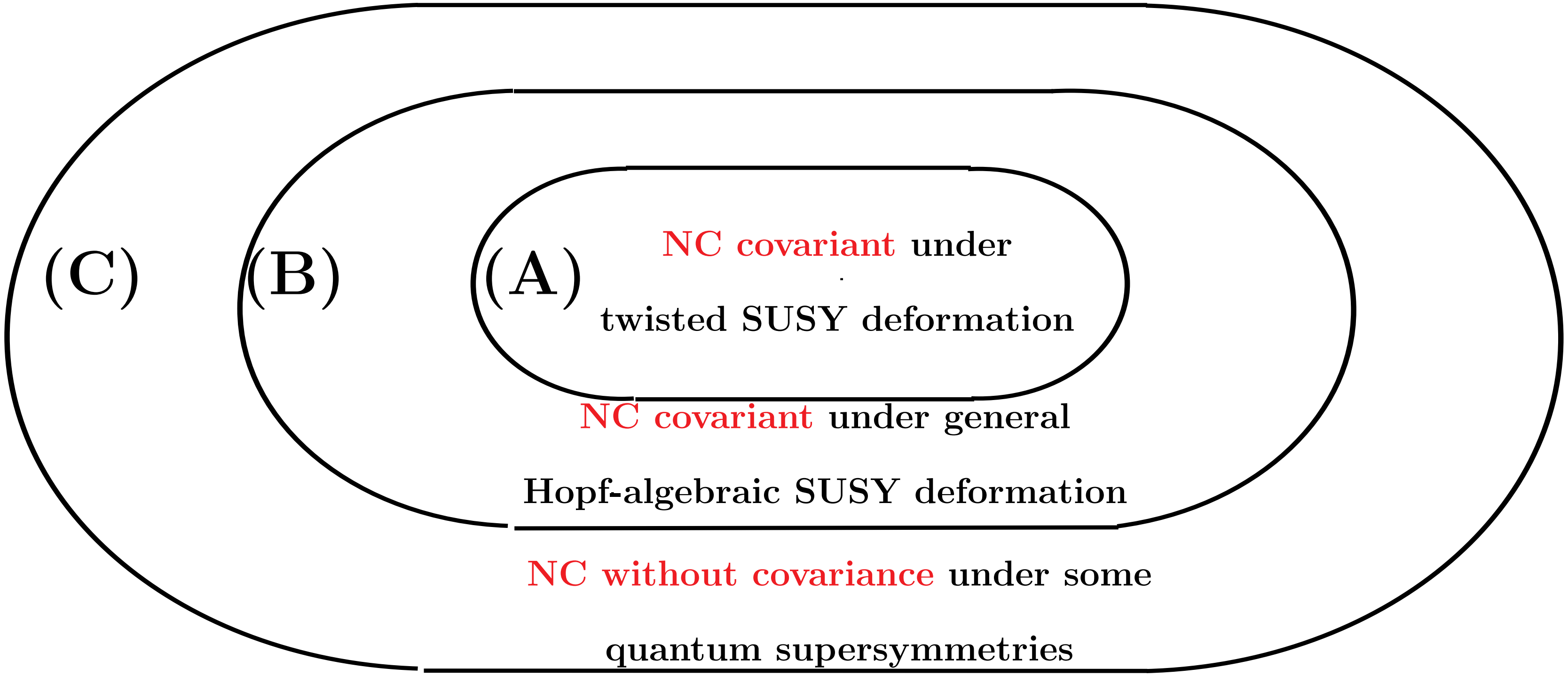
$$\begin{array}{ccc} \text{Theories of fundamental interactions supersymmetric} & \longrightarrow & \text{supersymmetric extension of quantum deformations} \\ \text{Lie algebra} & \xrightarrow{\kappa < \infty} & \text{quantum Lie algebra} \\ \downarrow \text{SUSY} & & \downarrow \text{SUSY} \\ \text{Lie superalgebra} & \xrightarrow[\kappa < \infty]{} & \text{quantum Lie superalgebra} \end{array}$$

Noncommutative superspace (the simplest chiral case)

$$\begin{array}{lcll} [x_\mu, x_\nu] = 0 & & [\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa^2}\theta_{\mu\nu}(\kappa\hat{x}, \kappa^{\frac{1}{2}}\theta) & \text{(even)} \\ [x_\mu, \theta_\alpha] = 0 & \xrightarrow[\kappa < \infty]{} & [\hat{x}_\mu, \hat{\theta}_\alpha] = \frac{i}{\kappa^{3/2}}\psi_{\mu\alpha}(\kappa\hat{x}, \kappa^{\frac{1}{2}}\theta) & \text{(odd)} \\ \{\theta_\alpha, \theta_\beta\} = 0 & & \{\hat{\theta}_\alpha, \hat{\theta}_\beta\} = \frac{i}{\kappa}C_{\alpha\beta}(\kappa\hat{x}, \kappa^{\frac{1}{2}}\theta) & \text{(even)} \end{array}$$

Problem: to select noncommutative superspaces which are covariant under quantum supersymmetries.

The structure of NC superspaces



Twisted deformations $\longleftrightarrow$ **infinitesimal deformations described by classical r-matrix satisfying classical YBE**

$\downarrow$ $\qquad\qquad$ $\downarrow$

supertwist deformations $\longleftrightarrow$ **infinitesimal deformations described by SUSY classical r-matrix satisfying classical SUSY YBE**

For Poincaré algebra (nonSUSY case):

All twisted deformations (A) and most of nontwisted quantum deformations were classified by providing explicite formulae for the classical r-matrices (S. Zakrzewski, 1996)

Two problems:

i) How to extend Zakrzewski classification for Euclidean case ($O(3,1) \to O(4)$)

ii) How to extend supersymmetrically Zakrzewski classification and provide also Euclidean supersymmetric counterpart

Remark: Knowledge of classical Poincaré r-matrices satisfying CYBE (infinitesimal deformation) permits to introduce the corresponding twist and define explicitly twisted Hopf algebra as finite deformation (Tolstoy 2008)

2. D=4 POINCARÉ AND EUCLIDEAN CLASSICAL r-MATRICES

a) ***Poincaré and Euclidean algebras.***

Poincaré Lie algebra – $P(3,1) = 0(3,1) \ltimes T(3,1)$
Euclidean Lie algebra – $E(4) = 0(4) \ltimes T(4)$

$$[M_{\mu\nu}, M_{\lambda\rho}] = i(\eta_{\nu\lambda}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\lambda} + \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\mu\lambda}M_{\nu\rho})$$

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu) \qquad [P_\mu, P_\nu] = 0$$

where $M_{\mu\nu} = -M_{\nu\mu}$ **and**

Poincaré case: $\eta_{\mu\nu} \equiv \eta^P_{\mu\nu} = (1, -1, -1, -1)$

Euclidean case: $\eta_{\mu\nu} \equiv \eta^E_{\mu\nu} = (-1, -1, -1, -1)$

Both algebras are real, with the reality conditions

$$M^+_{\mu\nu} = M_{\mu\nu} \qquad P^+_\mu = P_\mu$$

Both are two real forms of complex Lie algebra

$$IO(4:C) = O(4;C) \ltimes T(4;C)$$

In $O(3)$ basis

$$M_i = \varepsilon_{ijk} M_{jk} \qquad N_i = M_{0i} \qquad (i = 1, 2, 3)$$

$P(3,1)$ and $E(4)$ take the form

$$[M_i, M_j] = i\varepsilon_{ijk} M_k \qquad [M_i, P_j] = i\varepsilon_{ijk} P_k$$

$$[M_i, N_j] = i\varepsilon_{ijk} N_k \qquad [N_i, P_j] = -i\delta_{ij} P_0$$

$$[N_i, N_j] = \zeta\, i\varepsilon_{ijk} M_k \qquad [N_i, P_0] = \zeta\, i\, P_i$$

$$[P_\mu, P_\nu] = 0 \qquad [M_i, P_0] = 0$$

where $\zeta = -1$ for Poincaré and $\zeta = 1$ for Euclidean case.
One gets $\zeta = -1 \to \zeta = 1$ if we substitute (Euclideization):

$$M \to M_i \quad N_i \to iN_i \quad P_i \to P_i \quad P_0 \to iP_0$$

Both $O(3,1)$ and $O(4)$ can be written in the same way in canonical basis $(e_\pm, h, e'_\pm, h')$ of $O(4;C) = O_L(3;C) \oplus O_R(3;C)$

$$[h, e_\pm] = \pm e_\pm \qquad [e_+, e_-] = 2h$$

$$[h, e'_\pm] = \pm e'_\pm \qquad [h', e_-] = \pm e'_\pm \qquad [e_\pm, e'_\pm] = \pm 2h'$$

$$[h', e'_\pm] = \mp e_\pm \qquad [e'_+, e'_-] = -2h$$

where

$$h = \chi N_3 \qquad h' = iM_3$$

$$e_\pm = (\chi N_1 \pm iM_2) \qquad e'_\pm = (iM_1 \mp \chi N_2)$$

and

$\chi = -i$ – for Lorentz algebra

$\chi = 1$ – for Euclidean algebra

The difference appears only in the reality conditions

$$O(3,1): e_\pm^+ = -e_\pm \qquad (e'_\pm)^+ = -e'_\pm, \qquad h^+ = -h, \qquad (h')^+ = -h'$$

$$O(4): \quad e_\pm^+ = e_\mp \qquad (e'_\pm)^+ = -e'_\mp, \qquad h^+ = h, \qquad (h')^+ = -h'$$

We obtain the same form of Poincaré and Euclidean algebra if we use $\tilde{P}_i = P_i$, $\tilde{P}_0 = i\chi P_0$, where we recall that $\chi = -i$ for Poincaré and $\chi = 1$ for Euclidean case. If

$$\tilde{P}_1, \tilde{P}_2, \tilde{P}_\pm = \tilde{P}_0 \pm \tilde{P}_3$$

one obtains $\tilde{P}_1^+ = \tilde{P}_1$, $\tilde{P}_2^+ = \tilde{P}_2$ and

$$\tilde{P}_\pm^+ = \tilde{P}_\pm \quad \text{(Lorentz)} \qquad\qquad \tilde{P}_\pm^+ = \tilde{P}_\mp \quad \text{(Euclidean)}$$

i.e. in Euclidean case $\tilde{P}_\pm$ are not Hermitean (complex)

Important: because Zakrzewski derivation used canonical basis valid for $O(4;C) \ltimes T(4;C)$, the difference between Poincaré and Euclidean r-matrices is manifested only in different reality conditions.

c	b	a	$\mathcal{N}$
$\gamma h' \wedge h$	0	$\alpha P_+ \wedge P_- + \tilde{\alpha} P_1 \wedge P_2$	1
$\gamma e'_+ \wedge e_+$	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	2
	$\beta_1 b_{P_+}$	$\alpha P_+ \wedge P_1$	3
	$\gamma\beta_1(P_1 \wedge e_+ + P_2 \wedge e'_+)$	$P_+ \wedge (\alpha_1 P_1 + \alpha_2 P_2) - \gamma\beta_1^2 P_1 \wedge P_2$	4
$\gamma(h \wedge e_+$ $-h' \wedge e'_+)$ $+\gamma_1 e'_+ \wedge e_+$	0	0	5
$\gamma h \wedge e_+$	$\beta_1 b_{P_2} + \beta_2 P_2 \wedge e_+$	0	6
0	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge h'$	0	7
	$\beta_1 b_{P_+} + \beta_2 P_+ \wedge e_+$	0	8
	$P_1 \wedge (\beta_1 e_+ + \beta_2 e'_+) +$ $\beta_1 P_+ \wedge (h + \sigma e_+),\ \sigma = 0, \pm 1$	$\alpha P_+ \wedge P_2$	9
	$\beta_1(P_1 \wedge e'_+ + P_+ \wedge e_+)$	$\alpha_1 P_- \wedge P_1 + \alpha_2 P_+ \wedge P_2$	10
	$\beta_1 P_2 \wedge e_+$	$\alpha_1 P_+ \wedge P_1 + \alpha_2 P_- \wedge P_2$	11
	$\beta_1 P_+ \wedge e_+$	$P_- \wedge (\alpha P_+ + \alpha_1 P_1 + \alpha_2 P_2) + \tilde{\alpha} P_+ \wedge P_2$	12
	$\beta_1 P_0 \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	13
	$\beta_1 P_3 \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	14
	$\beta_1 P_+ \wedge h'$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	15
	$\beta_1 P_1 \wedge h$	$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	16
	$\beta_1 P_+ \wedge h$	$\alpha P_1 \wedge P_2 + \alpha_1 P_+ \wedge P_1$	17
	$P_+ \wedge (\beta_1 h + \beta_2 h')$	$\alpha_1 P_1 \wedge P_2$	18
	0	$\alpha_1 P_1 \wedge P_+$	19
		$\alpha_1 P_1 \wedge P_2$	20
		$\alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$	21

Classical D=4 Poincaré r-matrices

(Zakrzewski Table)

List of real classical r-matrices for D=4 Euclidean algebra:

Eight real classical r-matrices which do not contain complex generators P_+ (except $P_+ \wedge P_-$) and $e_\pm, e'_\pm$:

1. $r_1 = \gamma M_3 \wedge N_3 + \alpha(P_+ \wedge P_- + P_1 \wedge P_2)$ (1)
2. $r_2 = \beta_1 P_0 \wedge M_3 + \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$ (13)
3. $r_3 = \beta_2 P_3 \wedge M_3 + \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$ (14)
4. $r_4 = \beta_3(P_0 + P_3) \wedge M_3 + \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$ (15)
5. $r_5 = \beta_4(P_0 + P_3) \wedge N_3 + \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$ (17)
6. $r_6 = \beta_5 P_1 \wedge N_3 + \alpha_1 P_0 \wedge P_3 + \alpha_2 P_1 \wedge P_2$ (16)
7. $r_7 = \alpha_2 P_1 \wedge P_2$ (20)
8. $r_8 = \alpha_1(P_+ \wedge P_- + P_1 \wedge P_2)$ (21)

The r-matrices do not contain M_1, M_2 and N_1, N_2 - enters only Abelian subalgebra $O(2) \oplus O(2)$ of $O_L(3) \oplus O_R(3)$.

3. D=4 POINCARÉ AND EUCLIDEAN SUPERALGEBRAS

Let us write the complexified superalgebra with bosonic sector $O(4;C) \ltimes T^4(C)$ where $O(4,C) = O_L(3;C) \oplus O_R(3;C)$ and spinorial covering

$$\overline{O(4,C)} = Sl_L(2;C) \oplus Sl_R(2;C)$$

We introduce left and right supercharges ($\alpha = 1,2$)

$$Sl_L(2;C): \quad Q_{\alpha;}, \bar{Q}^{\dot{\alpha}}_{;} \qquad\qquad Sl_R(2;C): \quad Q^{\dot{\alpha}}_{;}, Q_{;\alpha}$$

with bilinear fermionic relations ($Q^+_{\alpha;} = \bar{Q}^{\dot{\alpha}}_{;}, (Q^{;\dot{\alpha}})^+ = \bar{Q}^{;}_{\alpha}$)

$$\{Q_{\alpha;}, \bar{Q}^{\dot{\beta}}_{;}\} = 2(\sigma^E_\mu)_{\alpha;}{}^{\dot{\beta}}\mathcal{P}^\mu \qquad \sigma_E{}^\mu = (\sigma_i, iI_2)$$

$$\{Q_{\alpha;}, Q_{;\beta}\} = \{Q^{\dot{\alpha}}_{;}, Q_{;}{}^{\dot{\beta}}\} = 0 \qquad \mathcal{P}^\mu \ - \ \text{complex}$$

Reality constraints for supercharges

a) real Poincaré algebra $\rightarrow$ N=1 superPoincaré algebra:

$$Q_{\alpha;} \equiv Q_\alpha \qquad\qquad Q_{;\dot{\alpha}} = Q^+_\alpha \ \leftrightarrow \ O(3,1) = O(3,C) \oplus \overline{O(3,C)}$$

$$\mathcal{P}_i = \mathcal{P}^+_i \qquad \mathcal{P}_0 = -\mathcal{P}^+_0$$

One gets ($P_\mu = (\mathcal{P}_i, i\mathcal{P}_0)$ – real Minkowski fourmomentum)

$$\{Q_\alpha, Q^+_{\dot\beta}\} = 2(\sigma_\mu)_{\alpha\dot\beta} P^\mu \qquad \sigma^\mu = (\sigma_i, I_2)$$

$$\{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0$$

$\Uparrow$ Minkowski choice

and Lorentz covariance relations

$$[M_{\mu\nu}, Q_\alpha] = -(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \qquad [M_{\mu\nu}, \bar Q_{\dot\alpha}] = \bar Q_{\dot\beta}(\bar\sigma_{\mu\nu})_\alpha{}^{\dot\beta}$$

where

$$\sigma_{\mu\nu} = \frac{1}{4}\sigma_{[\mu}\bar\sigma_{\nu]} \qquad \bar\sigma_{\mu\nu} = \frac{1}{4}\bar\sigma_{[\mu}\sigma_{\nu]}$$

b) real Euclidean algebra → holomorphic and antiholomorhic Euclidean superalgebras

Complex $Q_{\alpha;}$ and $Q_{;\alpha}$ remain independent.

For real Euclidean algebra $O(4;C) \to O(4;R) = O_L(3) \oplus O_R(3)$

and for spinors $Sl_L(2;C) \oplus Sl_R(2;C) \to SU_L(2) \oplus SU_R(2)$

For real Euclidean superalgebra the theory can be made covariant under the replacement which does not violate $SU_L(2)$ and $SU_R(2)$ covariance

$$Q^{\#}_{\alpha;} = \bar{Q}^{\dot{\alpha};} \equiv \varepsilon^{\dot{\alpha}\dot{\beta}}(Q_{\beta;})^{+} \qquad\qquad Q^{\#}_{;\alpha} = \bar{Q}^{;\dot{\alpha}} \equiv \varepsilon^{\dot{\alpha}\dot{\beta}}(Q_{;\beta})^{+}$$

Due to $(\varepsilon_{\alpha\beta})^2 = -1$ it is antilinear antiisomorphism of fourth order, so-called pseudoconjugation

$$(\hat{Q}^{\#})^{\#} = -1 \quad \rightarrow \quad Q^{\#}_{\alpha;} = Q_{\alpha;} \text{ is inconsistent}$$

The complexified superalgebra can be written in two forms: holomorphic and antiholomorphic, linked by pseudoconjugation: ($P^E_\mu = \mathcal{P}_\mu = \mathcal{P}^+_\mu$ – real Euclidean four-momenta)

$$\underset{\substack{\nearrow \\ \text{hol.}}}{\{Q_{\alpha;}}, \bar{Q}^{\dot{\beta}}_{;}\} = 2(\sigma^{\mu}_{E})_{\alpha;}{}^{\dot{\beta}}\, P^E_\mu \underset{\substack{\uparrow \\ \text{reality condition}}}{=} \{(Q_{\alpha;})^{\#}, (\bar{Q}_{;}{}^{\dot{\beta}})^{\#}\} \equiv \{\bar{Q}^{\dot{\alpha};}, \underset{\substack{\nwarrow \\ \text{antihol.}}}{Q^{;}_{\beta}\}}$$

$$\{Q_{\alpha;}, Q_{\beta;}\} = \{(Q_{\alpha;})^{\#}, (Q_{\beta;})^{\#}\} \equiv \{\bar{Q}^{\dot{\alpha};}, \bar{Q}^{\dot{\beta};}\} = 0$$

where the reality of Euclidean fourmomenta ${P_\mu}^E$ follows from

$$(\sigma^\mu_E)_{\alpha;}{}^{\dot\beta} = (\underset{\substack{\Uparrow\\ \text{Euclidean}}}{\sigma_i, iE}) \quad \Rightarrow \quad \sigma^{\mu\ \dot\alpha;}_{E\ \ \ \beta} = -\varepsilon^{\dot\alpha\dot\gamma}\left[(\sigma^\mu_E)_{\dot\gamma;}{}^\delta\right]^\star \varepsilon_{\delta\beta}$$

We have two types of complex N=1 Euclidean superalgebras:

1) holomorphic: $(Q_{\alpha;}, \bar Q^{;\dot\beta})$ ↖ linked by

2) antiholomorphic: $(Q_{\alpha;}, \bar Q^{\dot\beta;})$ ↙ pseudoconjugation #

N=1 holomorphic ⊕ N=1 antiholomorphic ⇒ selfconjugate N=2 real Euclidean superalgebra:

$$Q^i_{\alpha;}, Q^i_{;\alpha}, \quad (Q^i_{\alpha;})^+ = \bar Q^{\dot\alpha;}_i, \quad (Q^i_{;\alpha})^+ = \bar Q^{;\dot\alpha}_i \qquad i = 1, 2$$

Eight complex supercharges which due to $SU(2)$-Majorana condition are constrained to eight independent real supercharges

$$(Q^i_{\alpha;})^\# = \varepsilon_{ij} Q^j_{\alpha;} \;\Leftrightarrow\; Q^i_{\alpha;} = \varepsilon^{ij}\varepsilon_{\alpha\beta}(Q^j_{\alpha;})^+$$

4. D=4 POINCARÉ AND EUCLIDEAN CLASSICAL SUPERSYMMETRIC r-MATRICES

Non SUSY classical r-matrices $(L \equiv M_{\mu\nu}, P \equiv P_\mu)$

$$r_{\mathcal{P}}^{\mathrm{NONSUSY}} = r_{\mathrm{LL}} + r_{\mathrm{LP}} + r_{\mathrm{PP}}$$

$$r_{\mathrm{LL}} \in L \wedge L \qquad r_{\mathrm{LP}} \in L \wedge P \qquad r_{\mathrm{PP}} \in P \wedge P$$

General N=1 SUSY Poincaré r-matrices $(Q \equiv Q_\alpha, \bar{Q} \equiv \bar{Q}_{\dot{\alpha}})$:

$$r_{\mathcal{P}}^{\mathrm{SUSY}} = r_{\mathrm{NONSUSY}} + r_{\mathrm{QQ}}^{\mathrm{S}} + r_{Q\bar{Q}}^{\mathrm{S}} + r_{\bar{Q}\bar{Q}}^{\mathrm{S}}$$

It appears that the reality condition is satisfied only by the cases described by $r_{Q\bar{Q}}^{\mathrm{S}}$ $(Q_\alpha \wedge \bar{Q}_{\dot{\beta}} \equiv Q_\alpha \otimes \bar{Q}_{\dot{\beta}} + \bar{Q}_{\dot{\beta}} \otimes Q_\alpha)$

$$r_{Q\bar{Q}}^{\mathrm{S}} = r^{\alpha\beta} Q_\alpha \wedge \bar{Q}_{\dot{\beta}} \;\Rightarrow\; r^{\alpha\beta} = (r^{\beta\alpha})^\star$$

Seven out of 21 Poincaré r-matrices can be supersymmetrically extended (only one case contains $r_{\mathrm{QQ}}^{\mathrm{S}}$ and $r_{\bar{Q}\bar{Q}}^{\mathrm{S}}$)

List of seven supersymmetric classical Poincaré r-matrices: satisfying Poincaré reality condition

$$
\begin{aligned}
N=2:\quad & r_2 &&= \gamma e'_+ \wedge e_+ + \beta_1 b_{P_+} + \beta_2 P_+ \wedge h' + \beta_1 \bar{Q}_{\dot{1}} \wedge Q_1 \\
N=3:\quad & r_3 &&= \beta_1 b_{P_+} + \alpha P_+ \wedge P_1 + \beta_1 \bar{Q}_{\dot{1}} \wedge Q_1 \\
N=6:\quad & r_6 &&= \gamma h \wedge e_+ + \beta_1 b_{P_2} + \beta_2 P_2 \wedge e_+ + i\beta_1 (Q_1 + \bar{Q}_{\dot{1}} \wedge (Q_2 - \bar{Q}_{\dot{2}}) \\
N=7:\quad & r_7 &&= \beta_1 b_{P_+} + \beta_2 P_+ \wedge h' + \beta_1 \bar{Q}_{\dot{1}} \wedge Q_1 \\
N=8:\quad & r_8 &&= \beta_1 b_{P_+} + \beta_2 P_+ \wedge e_+ + \beta_1 \bar{Q}_{\dot{1}} \wedge Q_1 \\
N=9:\quad & r_9 &&= P_1 \wedge (\beta_1 e_+ + \beta_2 e'_+) + \beta_1 P_+ \wedge (h \pm e_+) + \\
& && \quad \alpha P_+ \wedge P_2 + \beta_1 \bar{Q}_{\dot{1}} \wedge Q_1 \\
N=17:\quad & r_{17} &&= \beta_1 P_+ \wedge h + \alpha_2 P_1 \wedge P_2 + \alpha_1 P_+ \wedge P_1 + \beta_1 \bar{Q}_{\dot{1}} \wedge Q_1
\end{aligned}
$$

N=1 Euclidean SUSY r-matrices $(Q^L_\alpha = Q_{\alpha;},\ \bar{Q}^R = \bar{Q}^{;\dot{\alpha}})$

$$r_E^{\text{SUSY}} = r_E^{\text{NONSUSY}} + r^{\text{S}}_{Q^L Q^L} + r^{\text{S}}_{Q^L \bar{Q}^R} + r^{\text{S}}_{\bar{Q}^R \bar{Q}^R} \quad \left(\begin{array}{c}\text{holomorphic} \\ \text{N=1 SUSY} \\ \text{or N=(1,0)}\end{array}\right)$$

All three possible SUSY terms are complex.

We have only 8 cases describing Euclidean classical r-matrices - for all these cases (N=1, 13–16, 19–21 in Zakrzewski table) do exist the superextension satisfying classical SUSY YB equation. All these deformations are described by $r^{\text{S}}_{Q^L Q^L}$, i.e. describe N=(1,0) SUSY deformations or chiral SUSY deformations ($\bar{Q}^R$ does not contribute). We have the following formulae:

$$N = 1,\ 13 - 16 \qquad r^{\text{S}}_{Q^L Q^L} = \eta Q_{2;} \wedge Q_{1;} = \eta Q_{1;} \wedge Q_{;2}$$

$$N = 19 - 21 \qquad r^{\text{S}}_{Q^L Q^L} = \eta^{\alpha\beta} Q_{\alpha;} \wedge Q_{\beta;}$$

$\leftarrow$ Superextension of canonical DFR deformation

Remark: For complex Euclidean SUSY r-matrices only in one out of 21 cases (N=5) we do not have supersymmetric extensions

5. FINAL REMARKS

a) For N=1 Euclidean SUSY N=$\frac{1}{2}$ complex deformations of supersymmetry corresponds to Seiberg's modification of SUSY obtained by chiral deformation of Grassmann algebra into Clifford algebra.

b) For N=2 selfdual Euclidean SUSY one can add to chiral N=$\frac{1}{2}$ deformation its complex conjugation → real deformation

$$C^{\alpha\beta} Q_{\alpha;} \wedge Q_{\beta;} \;\rightarrow\; C^{\alpha\beta}(Q_{\alpha;} \wedge Q_{\beta;} + \bar{Q}^{\dot{\alpha};} \wedge \bar{Q}^{\dot{\beta};})$$

Because $\{Q_{\alpha;}, Q^{\beta;}\} = 0 \Rightarrow$ classical SUSY YBE is valid.

c) The classification is not complete because does not contain all solutions of modified SUSY YBE, which can not be described as supersymmetric twist deformations.