Discretization of new Weyl group orbit functions

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Outline

1. **Short and long orbit functions**
   - Lie groups/Lie algebras
   - Sign homomorphisms
   - S^s- and S^l-functions
2. **Discretization of orbit functions**
   - Grids F^s_M and F^l_M
   - Grids Λ^s_M and Λ^l_M
   - Discrete orthogonality of S^s- and S^l-functions

- R. V. Moody, L. Motlochová, and J. Patera, *New families of Weyl group orbit functions*, arXiv:1202.4415
- J. Hrivnák, L. Motlochová, J. Patera, *On discretization of tori of compact simple Lie groups II,* J. Phys. A: Math. Theor. **45** (2012) 255201, arXiv:1206.0240

- J. Hrivnák, J. Patera, *On discretization of tori of compact simple Lie groups,* J. Phys. A: Math. Theor. **42** (2009) 385208
- R. V. Moody, J. Patera, *Orthogonality within the families of C-, S-, and E- functions of any compact semisimple Lie group*, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) 2 (2006) 076, 14 pages, math-ph/0611020
- A. Klimyk, J. Patera, *Antisymmetric orbit functions*, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) 3 (2007), paper 023, 83 pages; math-ph/0702040
- A. Klimyk, J. Patera, *E-orbit functions*, SIGMA (Symmetry,Integrability and Geometry: Methods and Applications) 4 (2008), 002, 57 pages; arXiv:0801.0822

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Lie groups/Lie algebras

- the Lie algebra of the compact simply connected simple Lie group G of rank n
- the set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$, $\operatorname{span}_{\mathbb{R}} \Delta = \mathbb{R}^n$
- with two different lengths of the roots

$$\Delta = \Delta_s \cup \Delta_l$$

- $B_n \ (n \geq 3)$, $C_n \ (n \geq 2)$, F_4, G_2
- the highest root $\xi \equiv -\alpha_0 = m_1\alpha_1 + \dots + m_n\alpha_n$
- $m_j \dots$ the **marks** of G
- the Cartan matrix C

$$C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \quad i, j \in \{1, \dots, n\}$$

- and its determinant $c = \det C$

Root and weight lattices

- the root lattice Q of G

$$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$$

- the $\mathbb{Z}$-dual lattice to Q

$$P^\vee = \{\omega^\vee \in \mathbb{R}^n \,|\, \langle \omega^\vee, \alpha \rangle \in \mathbb{Z}, \forall \alpha \in \Delta\} = \mathbb{Z}\omega_1^\vee + \cdots + \mathbb{Z}\omega_n^\vee$$

- the dual root lattice

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee \,, \quad \text{where} \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$$

- the $\mathbb{Z}-$dual lattice to $Q^\vee$

$$P = \{\omega \in \mathbb{R}^n \,|\, \langle \omega, \alpha^\vee \rangle \in \mathbb{Z}, \forall \alpha^\vee \in \Delta^\vee\} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$$

Weyl group and affine Weyl group

- the Weyl group W is generated by n reflections r_α, $\alpha \in \Delta$

$$r_{\alpha_i} a \equiv r_i a = a - \frac{2\langle a,\, \alpha_i\rangle}{\langle \alpha_i,\, \alpha_i\rangle}\alpha_i\,, \qquad a \in \mathbb{R}^n$$

- W^{aff} is generated by the reflections r_i, $i \in \{1, \ldots, n\}$ and the reflection r_0

$$r_0 a = r_\xi a + \frac{2\xi}{\langle \xi,\, \xi\rangle}\,, \qquad r_\xi a = a - \frac{2\langle a,\, \xi\rangle}{\langle \xi,\, \xi\rangle}\xi\,, \qquad a \in \mathbb{R}^n$$

The affine Weyl group

$$W^{\mathrm{aff}} = Q^\vee \rtimes W$$

Long and short reflections

- W^{aff} is generated by $n+1$ reflections

$$R=\{r_0,r_1,\dots,r_n\}$$

- a disjoint decomposition $R=R^s\cup R^l$

$$R^s=\{r_\alpha\,|\,\alpha\in\Delta_s\}$$
$$R^l=\{r_\alpha\,|\,\alpha\in\Delta_l\}\cup\{r_0\}$$

- the short and the long Coxeter numbers

$$m^s=\sum_{\alpha_i\in\Delta_s}m_i,\quad m^l=\sum_{\alpha_i\in\Delta_l}m_i+1$$

Type	R^s	R^l	m^s	m^l
$B_n\ (n\geq 3)$	r_n	$r_0,\,r_1,\dots,r_{n-1}$	2	$2n-2$
$C_n\ (n\geq 2)$	$r_1,\dots,r_{n-1}$	$r_0,\,r_n$	$2n-2$	2
G_2	r_2	$r_0,\,r_1$	3	3
F_4	r_3,r_4	$r_0,\,r_1,\,r_2$	6	6

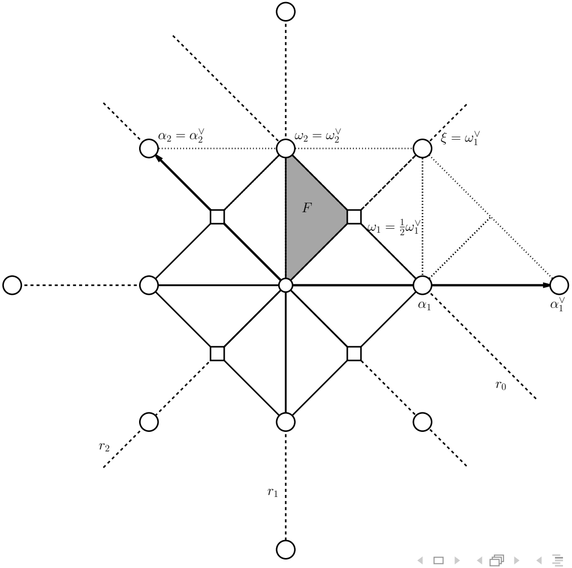
The fundamental domain

- domain in $\mathbb{R}^n$ which contains precisely one point from each W^{aff} orbit
- The fundamental region F of W^{aff}

$$F = \left\{ y_1\omega_1^\vee + \dots + y_n\omega_n^\vee \,\middle|\, y_0, \dots, y_n \in \mathbb{R}_0^+,\ y_0 + y_1 m_1 + \dots + y_n m_n = 1 \right\}$$
$$= \left\{ a \in \mathbb{R}^n \,\middle|\, \langle a,\, \alpha\rangle \geq 0, \forall \alpha \in \Delta, \langle a,\, \xi\rangle \leq 1 \right\}$$

$$F = \left\{ 0, \frac{\omega_1^\vee}{m_1}, \dots, \frac{\omega_n^\vee}{m_n} \right\}_\kappa$$

The fundamental domain F of C_2



Outline

Sign homomorphisms

- an abstract presentation of W

$$r_i^2 = 1, \quad (r_i r_j)^{m_{ij}} = 1, \quad i,j = 1,\dots,n$$

- m_{ij} are elements of the Coxeter matrix.
- 'sign' homomorphisms $\sigma : W \to \{\pm 1\}$

$$\sigma(r_i)^2 = 1, \quad (\sigma(r_i)\sigma(r_j))^{m_{ij}} = 1, \quad i,j = 1,\dots,n$$

- the four sign homomorphisms $\mathbb{1}$, σ^e, σ^s, σ^l:

$$\mathbb{1}(r_\alpha) = 1$$
$$\sigma^e(r_\alpha) = -1$$
$$\sigma^s(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_l \\ -1, & \alpha \in \Delta_s \end{cases}$$
$$\sigma^l(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_s \\ -1, & \alpha \in \Delta_l \end{cases}$$

Fundamental domains

- two subsets of boundaries of F:

$$H^s = \{a \in F \,|\, (\exists r \in R^s)(ra = a)\}$$
$$H^l = \left\{a \in F \,|\, (\exists r \in R^l)(ra = a)\right\}$$

- fundamental domains $F^s \subset F$, $F^l \subset F$

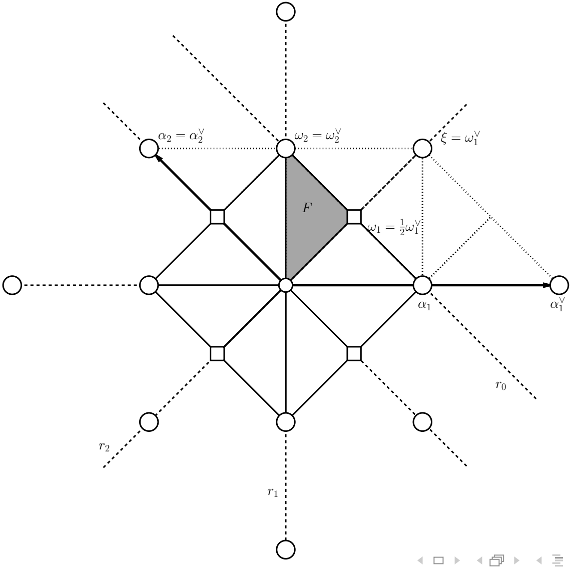
$$F^s = F \setminus H^s$$
$$F^l = F \setminus H^l$$

- the symbols y_i^s, $y_i^l \in \mathbb{R}$, $i = 0, \dots, n$

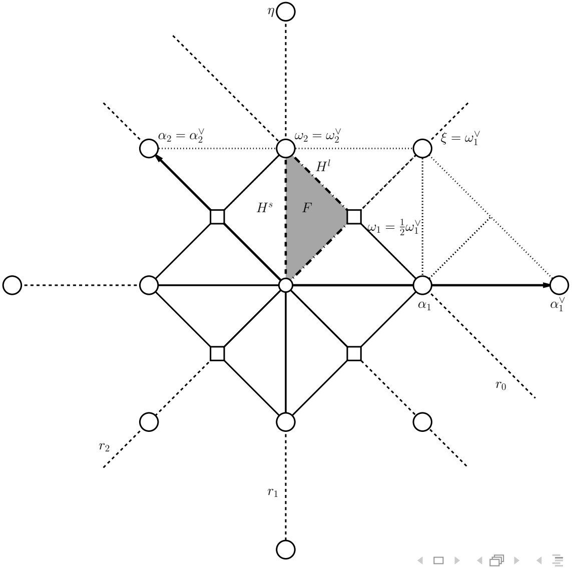
$$y_i^s > 0, \quad y_i^l \geq 0, \quad r_i \in R^s$$
$$y_i^s \geq 0, \quad y_i^l > 0, \quad r_i \in R^l$$

$$F^s = \left\{ y_1^s \omega_1^\vee + \dots + y_n^s \omega_n^\vee \,\middle|\, y_0^s + y_1^s m_1 + \dots + y_n^s m_n = 1 \right\}$$
$$F^l = \left\{ y_1^l \omega_1^\vee + \dots + y_n^l \omega_n^\vee \,\middle|\, y_0^l + y_1^l m_1 + \dots + y_n^l m_n = 1 \right\}$$

The fundamental domain F of C_2



The fundamental domains F^s and F^l of C_2



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S^s- and S^l-functions

- for $\sigma \in \{1, \sigma^e, \sigma^s, \sigma^l\}$, $b \in P$ are the complex functions $\varphi_b^\sigma : \mathbb{R}^n \to \mathbb{C}$

$$\varphi_b^\sigma(a) = \sum_{w \in W} \sigma(w)\, e^{2\pi i \langle wb,\, a\rangle}, \quad a \in \mathbb{R}^n$$

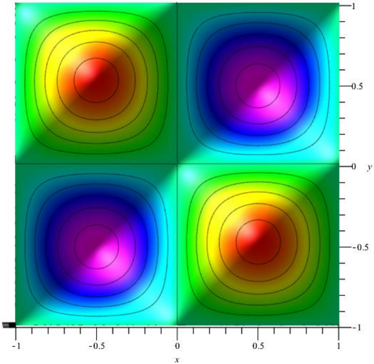
- $\sigma = \sigma^e \dots$ $S-$functions (known from the Weyl character formula)
- $\sigma = 1 \dots$ $C-$functions
- $\sigma = \sigma^s \dots$ S^s-functions
- $\sigma = \sigma^l \dots$ S^l-functions
- (anti)symmetry with respect to $w \in W$

$$\varphi_b^s(wa) = \sigma^s(w)\varphi_b^s(a)$$
$$\varphi_{wb}^s(a) = \sigma^s(w)\varphi_b^s(a)$$

- invariance with respect to shifts from $q^\vee \in Q^\vee$

$$\varphi_b^s(a + q^\vee) = \varphi_b^s(a)$$

S^s-function $\varphi^s_{(1,0)}(x,y)$ of C_2



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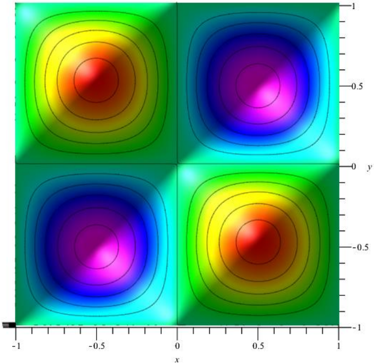

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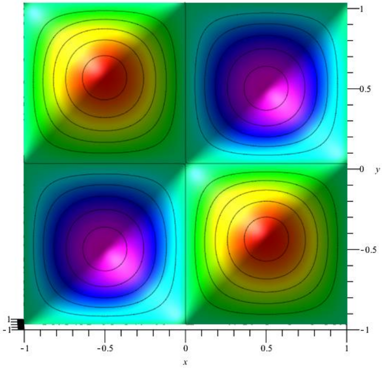

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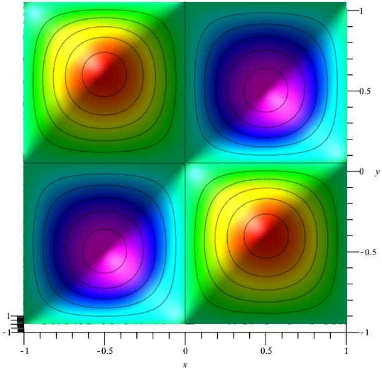

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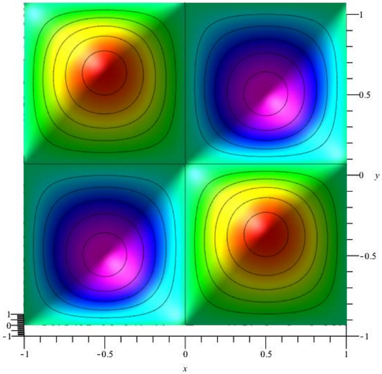

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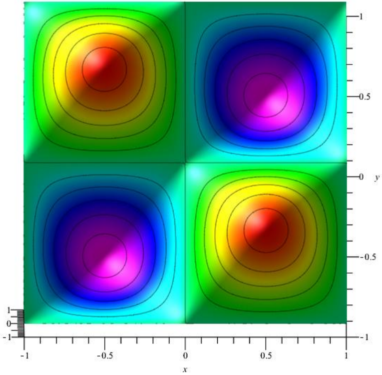

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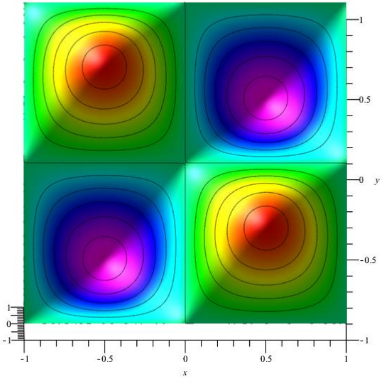

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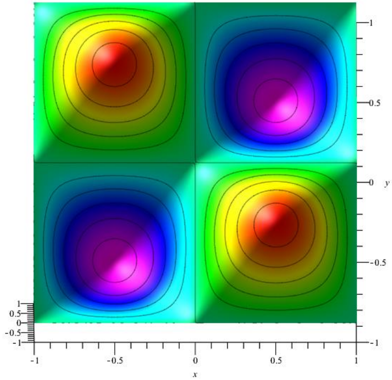

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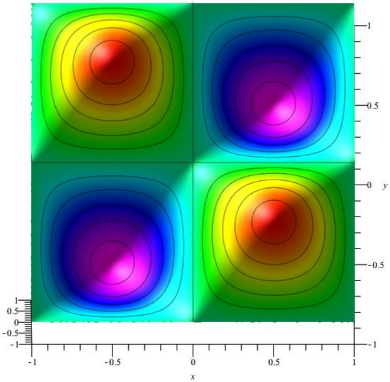

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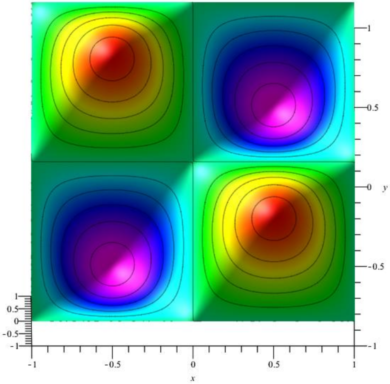

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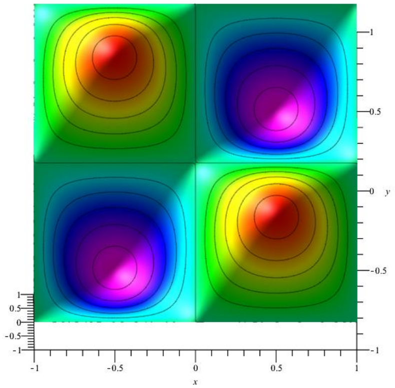

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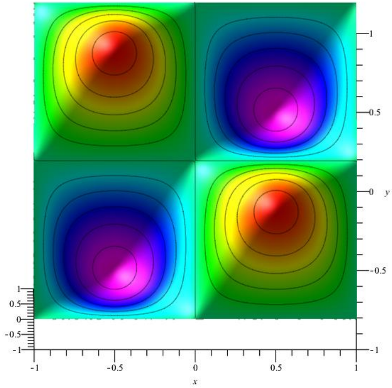

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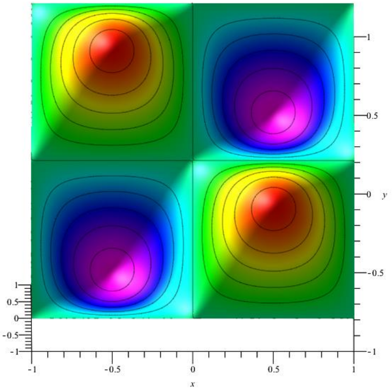

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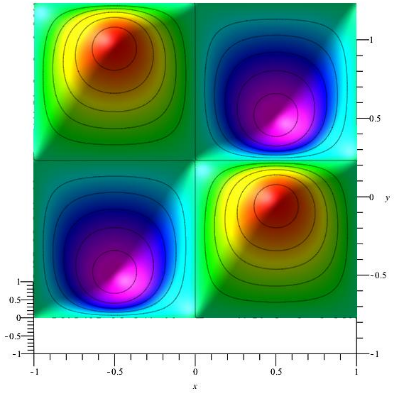

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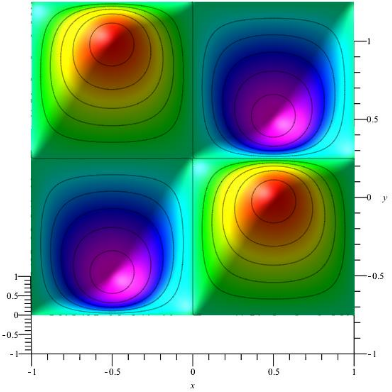

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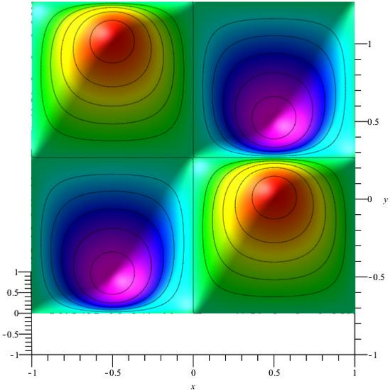

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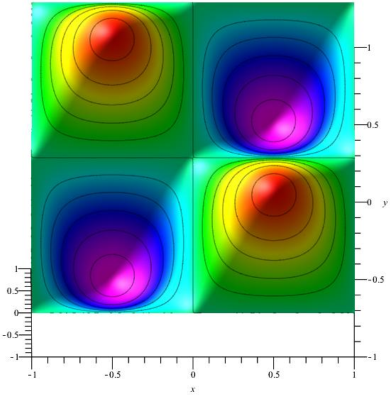

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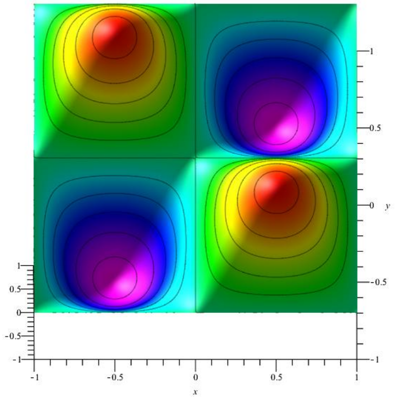

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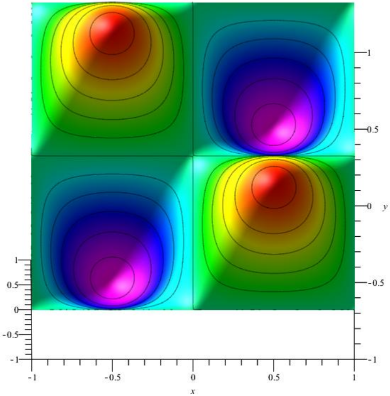

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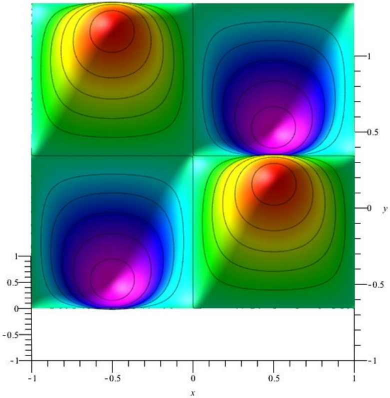

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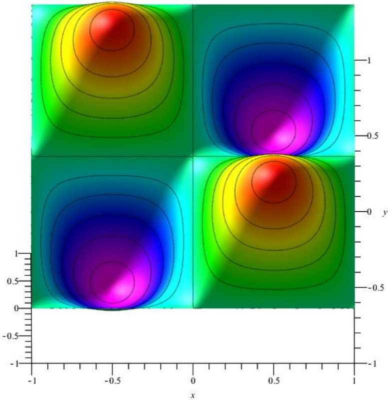

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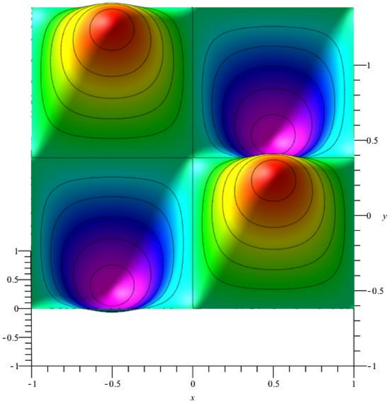

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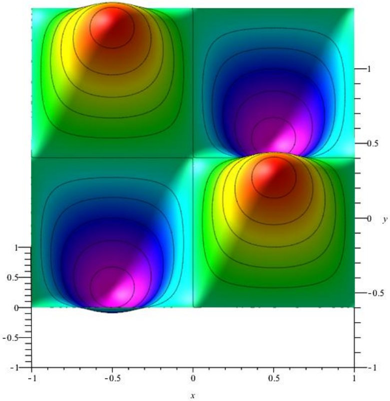

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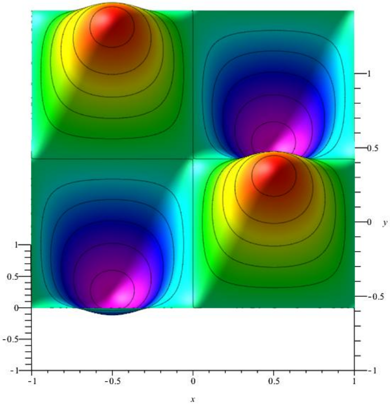

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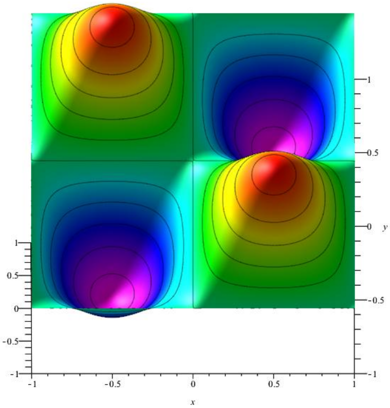

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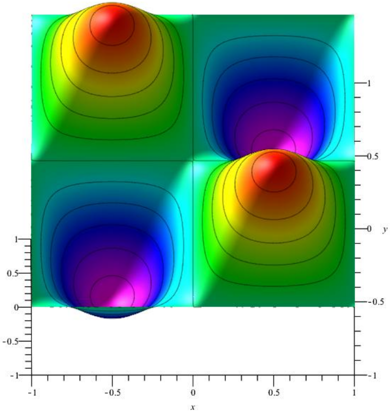

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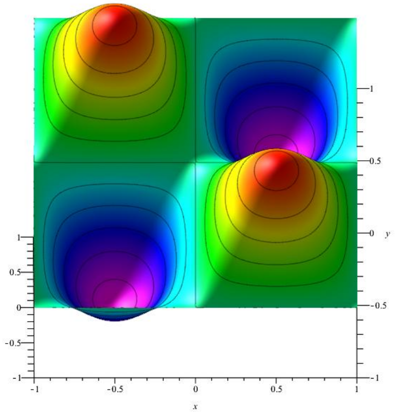

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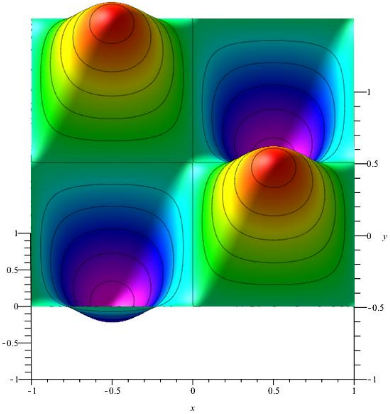

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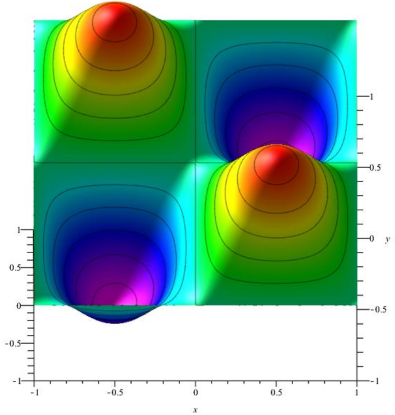

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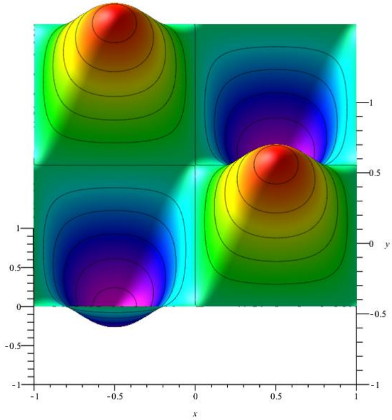

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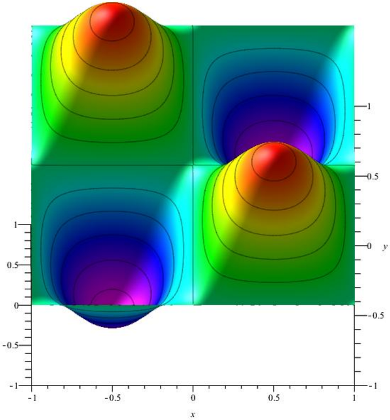

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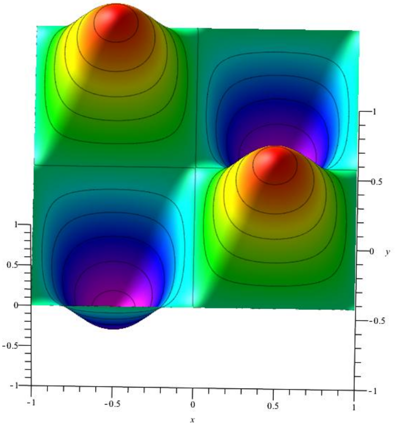

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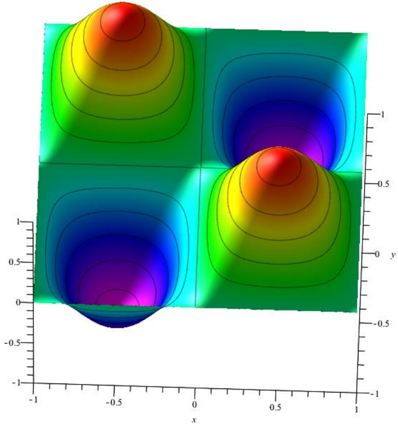

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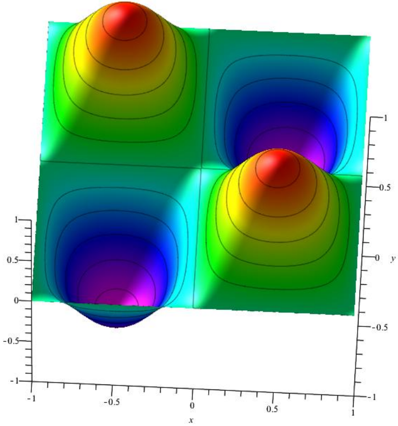

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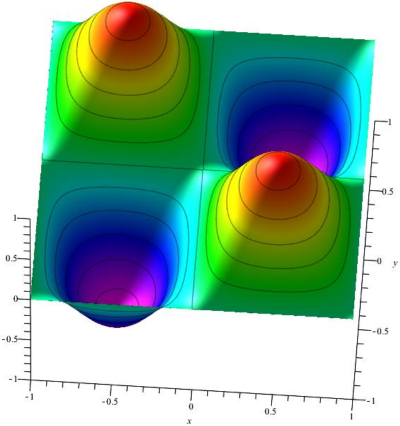

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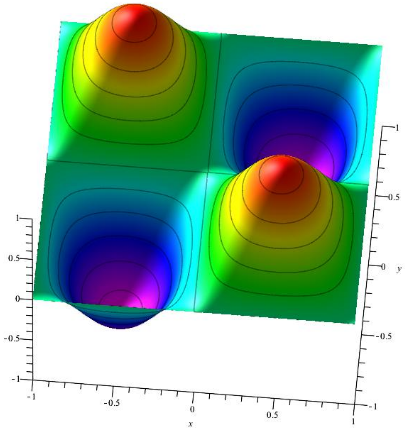

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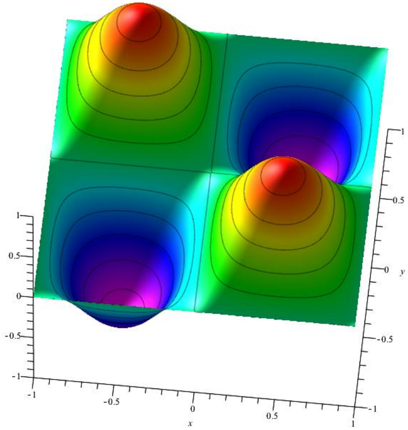

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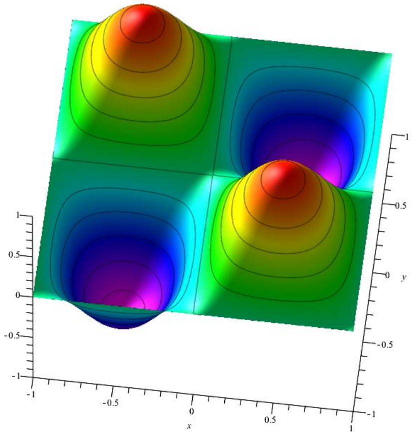
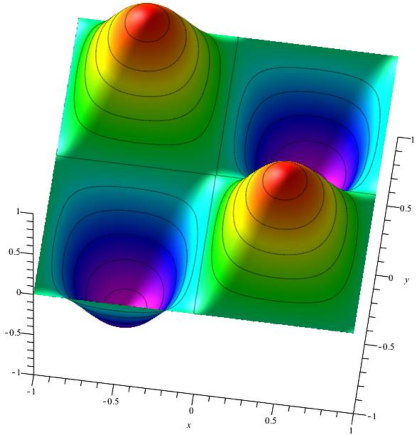


S^s-function $\varphi^s_{(1,0)}(x,y)$ of C_2



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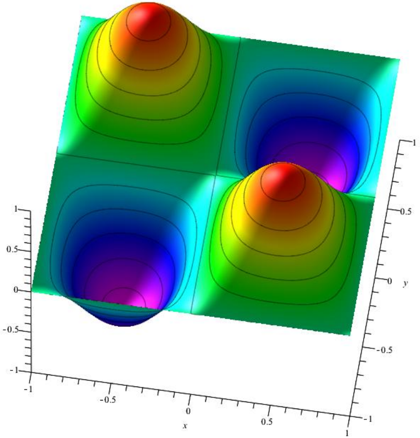
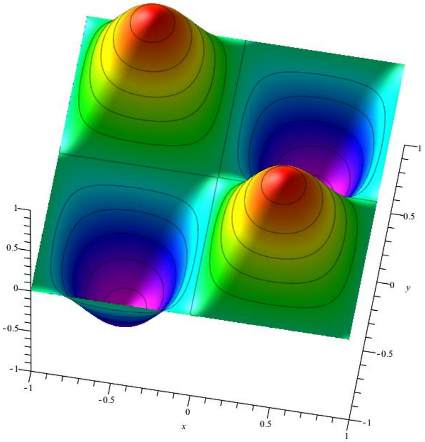
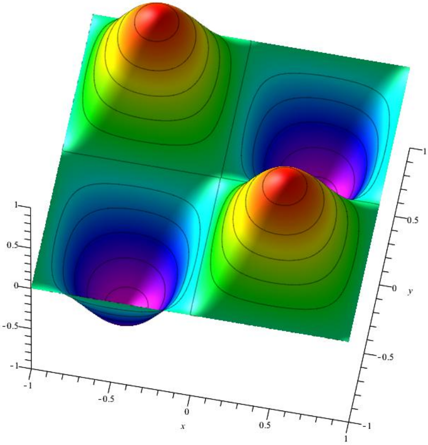


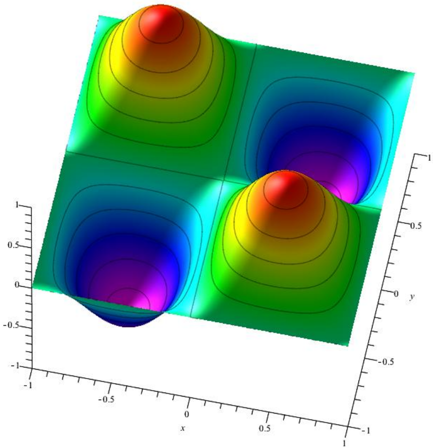
S^s-function $\varphi^s_{(1,0)}(x,y)$ of C_2



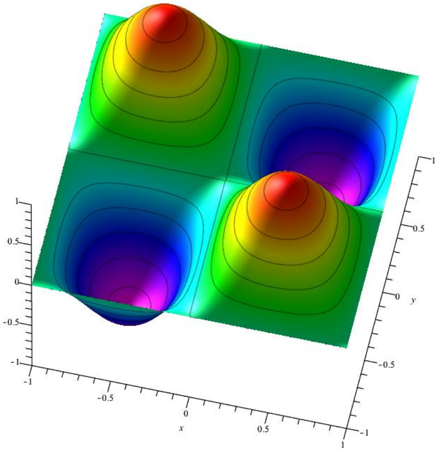
S^s-function $\varphi^s_{(1,0)}(x,y)$ of C_2



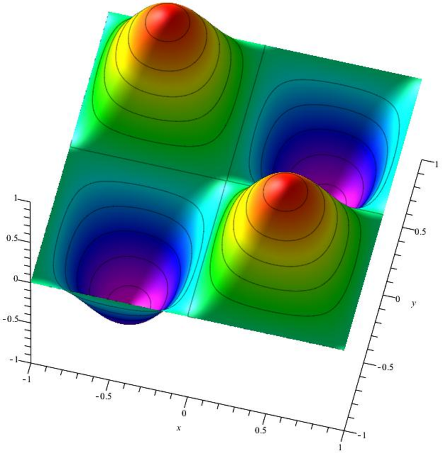
S^s–function $\varphi^s_{(1,0)}(x,y)$ of C_2



S^s-function $\varphi^s_{(1,0)}(x,y)$ of C_2



S^s-function $\varphi^s_{(1,0)}(x,y)$ of C_2



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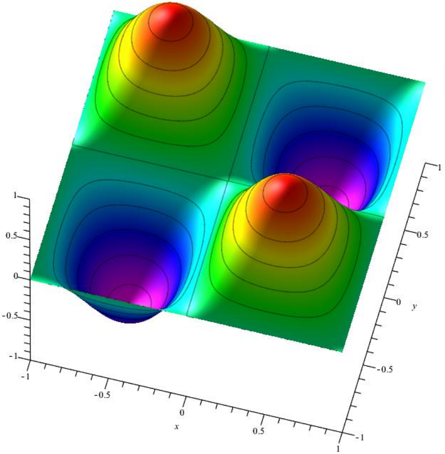
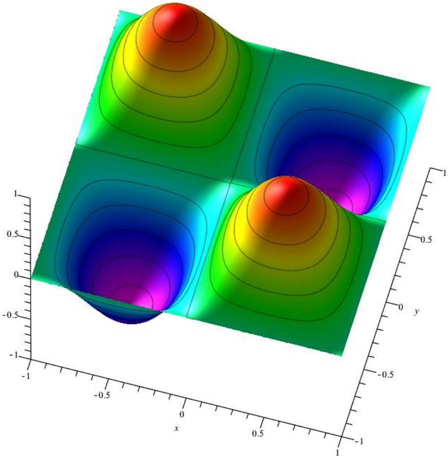
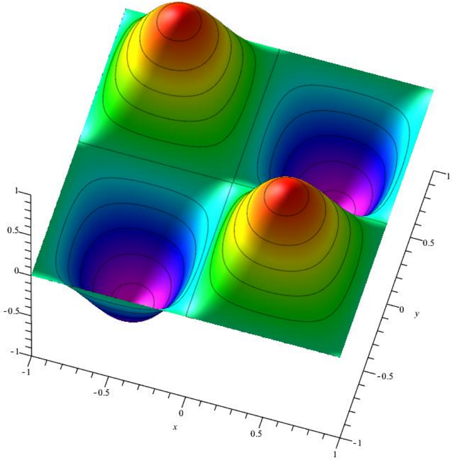


S^s-function $\varphi^s_{(1,0)}(x,y)$ of C_2



S^s-function $\varphi^s_{(1,0)}(x,y)$ of C_2



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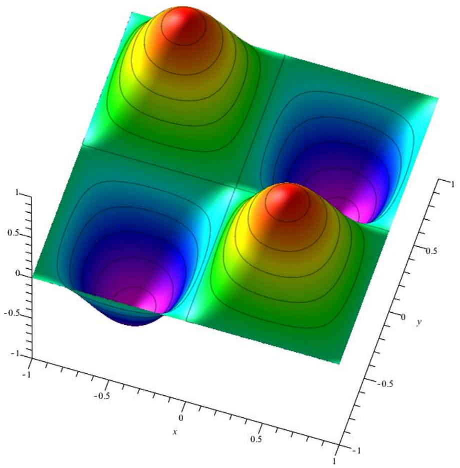
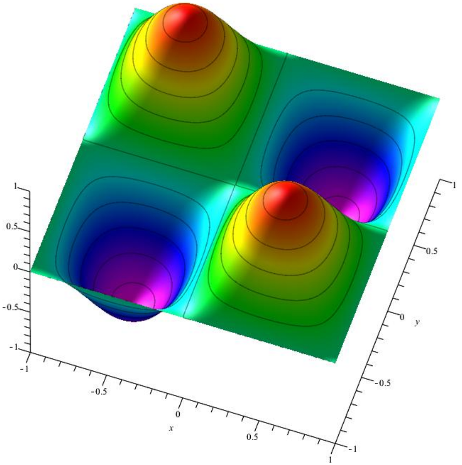


S^s−function $\varphi^s_{(1,0)}(x,y)$ of C_2



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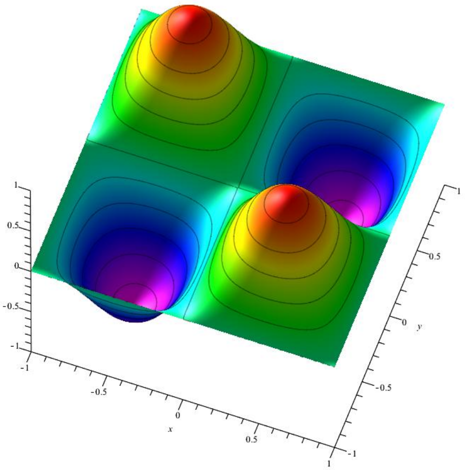

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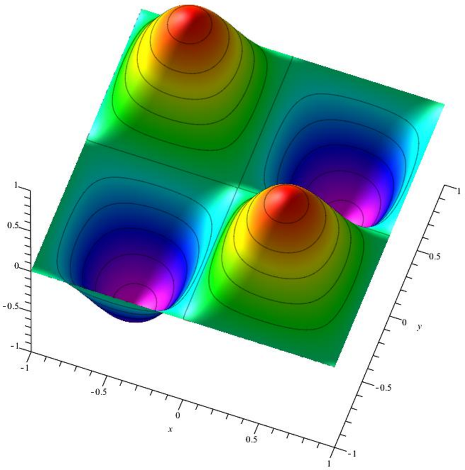

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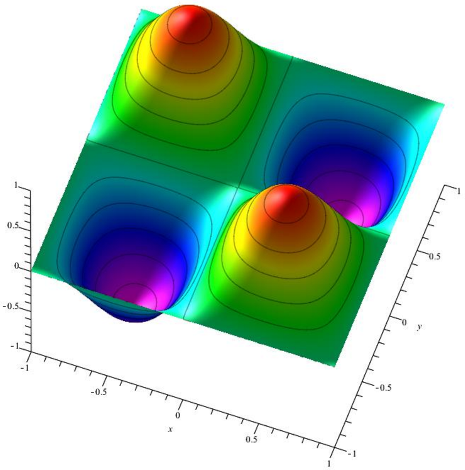
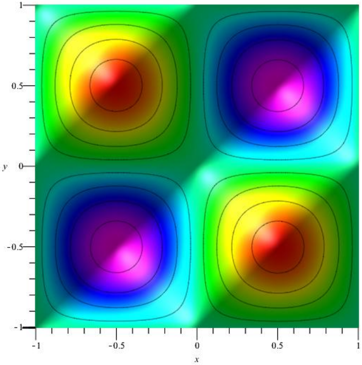
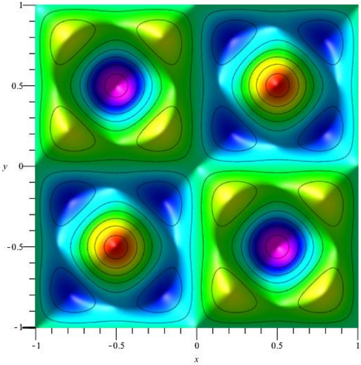
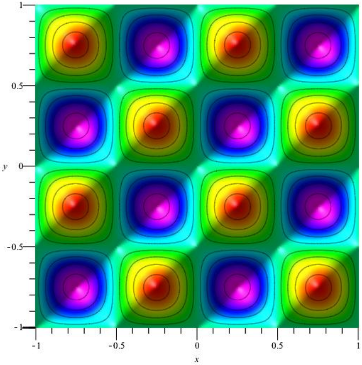


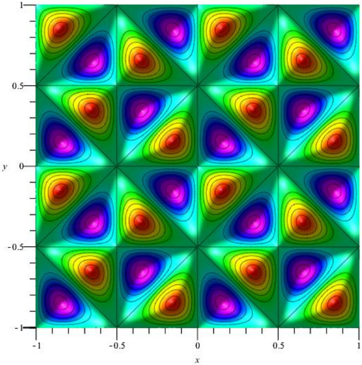
S^s-functions $\varphi^s_{(1,0)}(x,y)$ and $\varphi^s_{(1,1)}(x,y)$ of C_2



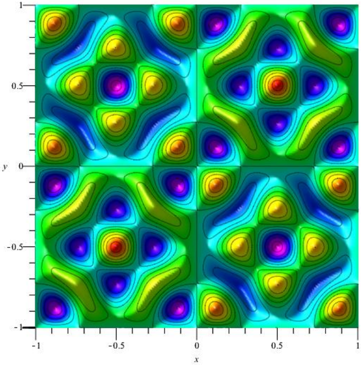


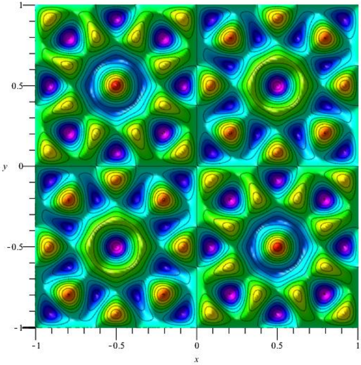
S^s-functions $\varphi^s_{(2,0)}(x,y)$ and $\varphi^s_{(2,1)}(x,y)$ of C_2



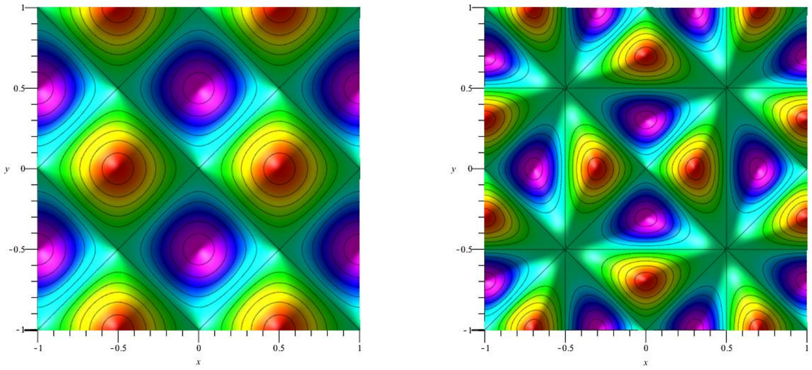


S^s-functions $\varphi^s_{(3,1)}(x,y)$ and $\varphi^s_{(3,2)}(x,y)$ of C_2

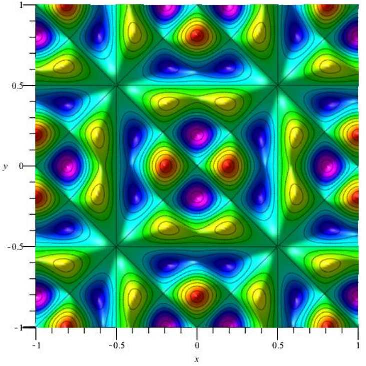


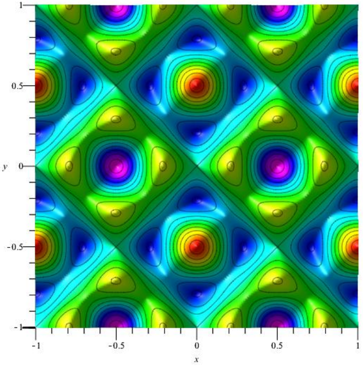


S^l-functions $\varphi^l_{(0,1)}(x,y)$ and $\varphi^l_{(1,1)}(x,y)$ of C_2

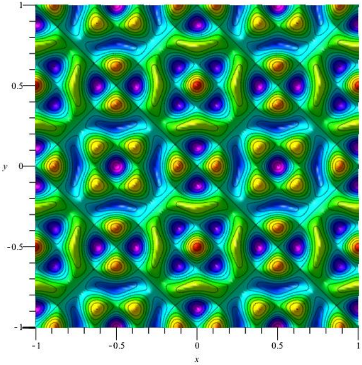


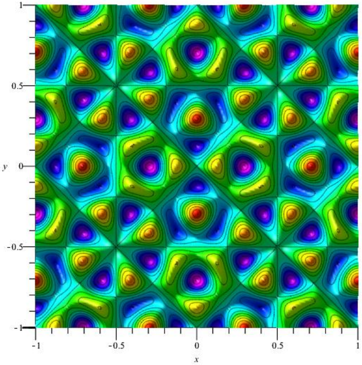
S^l-functions $\varphi^l_{(1,2)}(x,y)$ and $\varphi^l_{(2,1)}(x,y)$ of C_2



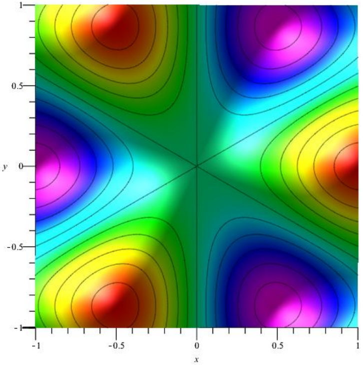


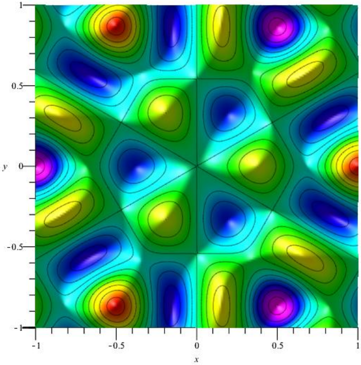
S^l-functions $\varphi^l_{(2,3)}(x,y)$ and $\varphi^l_{(3,2)}(x,y)$ of C_2



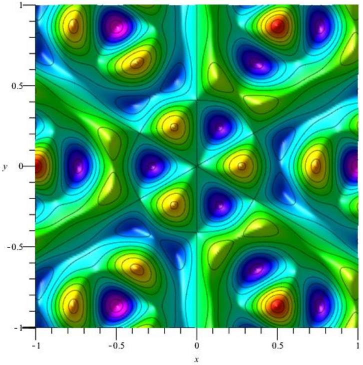


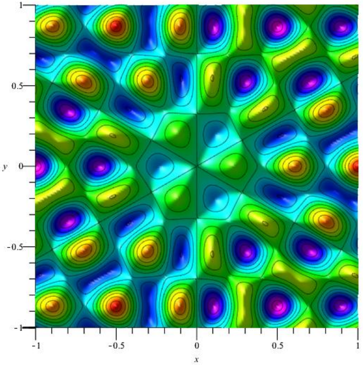
S^s-functions $\varphi^s_{(0,1)}(x,y)$ and $\varphi^s_{(1,1)}(x,y)$ of G_2



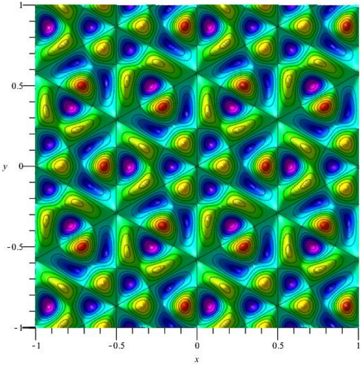


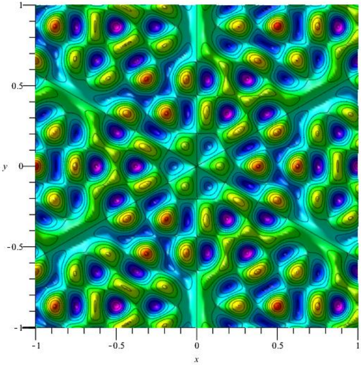
S^s-functions $\varphi^s_{(1,2)}(x,y)$ and $\varphi^s_{(2,1)}(x,y)$ of G_2



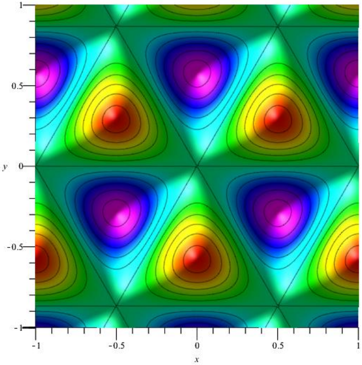


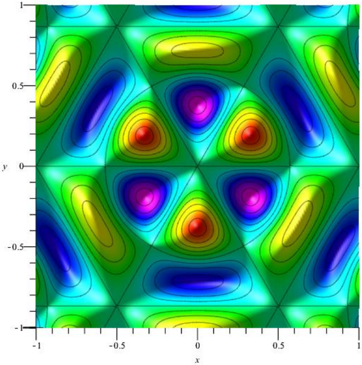
S^s-functions $\varphi^s_{(2,3)}(x,y)$ and $\varphi^s_{(3,2)}(x,y)$ of G_2



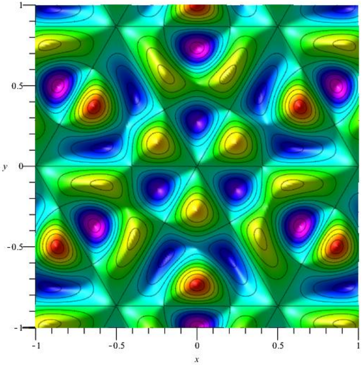


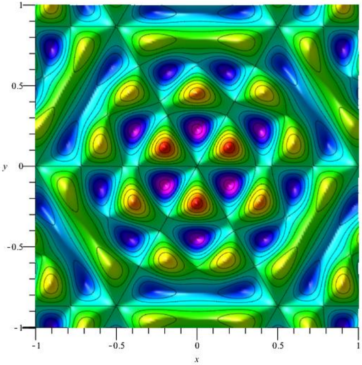
S^l-functions $\varphi^l_{(1,0)}(x,y)$ and $\varphi^l_{(1,1)}(x,y)$ of G_2



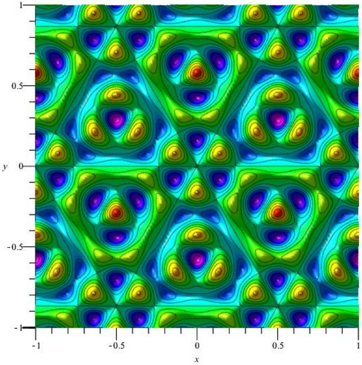


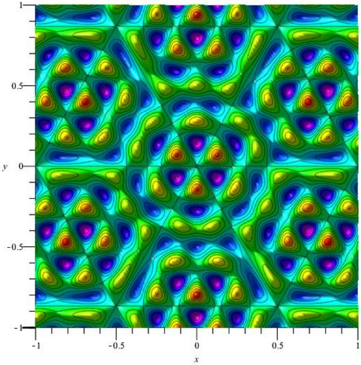
S^l-functions $\varphi^l_{(1,2)}(x,y)$ and $\varphi^l_{(2,1)}(x,y)$ of G_2





S^l-functions $\varphi^l_{(2,3)}(x,y)$ and $\varphi^l_{(3,2)}(x,y)$ of G_2





Outline

1. Short and long orbit functions
   - Lie groups/Lie algebras
   - Sign homomorphisms
   - S^s- and S^l-functions
2. **Discretization of orbit functions**
   - **Grids F_M^s and F_M^l**
   - Grids Λ_M^s and Λ_M^l
   - Discrete orthogonality of S^s- and S^l-functions

Grids F^s_M and F^l_M

- W-invariant lattice $\frac{1}{M}P^\vee$
- $M \in \mathbb{N}$, W-invariant finite group $\frac{1}{M}P^\vee/Q^\vee$
- number of elements of $\frac{1}{M}P^\vee/Q^\vee$ is cM^n

The grid F_M

$$F_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F$$

The grids $F^s_M \subset F_M$ and $F^l_M \subset F_M$

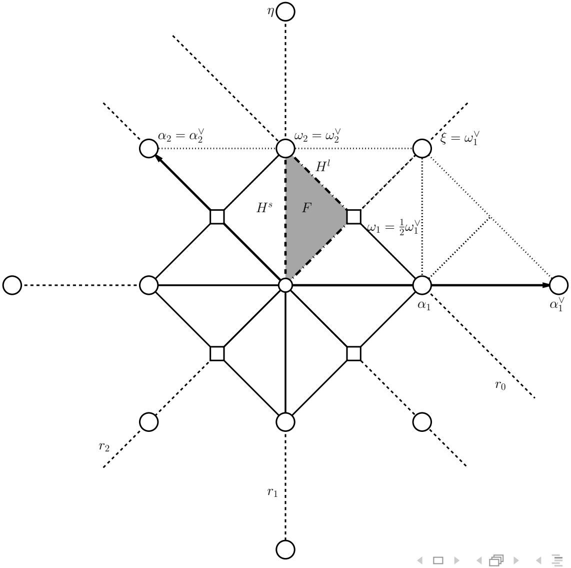
$$F^s_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F^s$$

$$F^l_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F^l$$

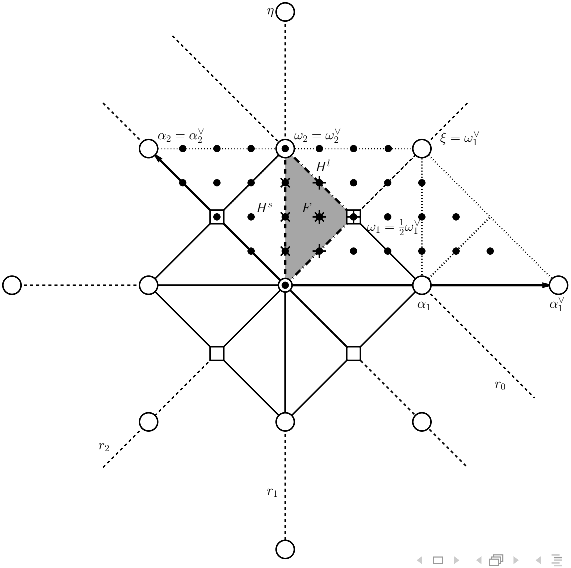
- the symbols $u^s_i,\, u^l_i \in \mathbb{R}$, $i = 0, \dots, n$:

$$u^s_i \in \mathbb{N}, \quad u^l_i \in \mathbb{Z}^{\geq 0}, \quad r_i \in R^s$$
$$u^s_i \in \mathbb{Z}^{\geq 0}, \quad u^l_i \in \mathbb{N}, \quad r_i \in R^l$$

Grids F^s_4 and F^l_4 of C_2



Grids F^s_4 and F^l_4 of C_2



Grids F_M^s and F_M^l

- the explicit form of F^s and F^l:

$$F_M^s = \left\{ \frac{u_1^s}{M}\omega_1^\vee + \dots + \frac{u_n^s}{M}\omega_n^\vee \,\middle|\, u_0^s + u_1^s m_1 + \dots + u_n^s m_n = M \right\}$$

$$F_M^l = \left\{ \frac{u_1^l}{M}\omega_1^\vee + \dots + \frac{u_n^l}{M}\omega_n^\vee \,\middle|\, u_0^l + u_1^l m_1 + \dots + u_n^l m_n = M \right\}$$

Proposition

Let m^s and m^l be the short and long Coxeter numbers, respectively. Then

$$|F_M^s| = \begin{cases} 0 & M < m^s \\ 1 & M = m^s \\ |F_{M-m^s}| & M > m^s. \end{cases}, \quad |F_M^l| = \begin{cases} 0 & M < m^l \\ 1 & M = m^l \\ |F_{M-m^l}| & M > m^l. \end{cases}$$

Grids F^s_M and F^l_M

Theorem

The numbers of points of grids F^s_M and F^l_M of Lie algebras B_n, C_n are given by the following relations.

1. C_n, $n \geq 2$,

$$|F^s_{2k}(C_n)| = \binom{k+1}{n} + \binom{k}{n}$$
$$|F^s_{2k+1}(C_n)| = 2\binom{k+1}{n}$$

$$|F^l_{2k}(C_n)| = \binom{n+k-1}{n} + \binom{n+k-2}{n}$$
$$|F^l_{2k+1}(C_n)| = 2\binom{n+k-1}{n}$$

2. B_n, $n \geq 3$,

$$|F^s_M(B_n)| = |F^l_M(C_n)|$$
$$|F^l_M(B_n)| = |F^s_M(C_n)|$$

Grids F^s_M and F^l_M

Theorem

The numbers of points of grids F^s_M and F^l_M of Lie algebra G_2 are given by the following relations.

$$|F^s_{6k}(G_2)| = 3k^2, \qquad |F^s_{6k+1}(G_2)| = 3k^2 + k$$
$$|F^s_{6k+2}(G_2)| = 3k^2 + 2k, \qquad |F^s_{6k+3}(G_2)| = 3k^2 + 3k + 1$$
$$|F^s_{6k+4}(G_2)| = 3k^2 + 4k + 1, \qquad |F^s_{6k+5}(G_2)| = 3k^2 + 5k + 2.$$

$$|F^l_M(G_2)| = |F^s_M(G_2)|$$

- F_4: the counting formula is also known

Outline

The dual Lie algebra

- $\Delta \to \Delta^\vee$:
$$\alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}, \quad i \in \{1, \dots, n\}$$
- the set $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ is a system of simple roots of some simple Lie algebra
$$\operatorname{span}_{\mathbb{R}} \Delta^\vee = \mathbb{R}^n$$
- the highest dual root $\eta \equiv -\alpha_0^\vee = m_1^\vee \alpha_1^\vee + \dots + m_n^\vee \alpha_n^\vee$
- $m_j^\vee$ ... the **dual marks** of G
- the dual Cartan matrix $C^\vee$
$$C_{ij}^\vee = \frac{2\langle \alpha_i^\vee, \alpha_j^\vee \rangle}{\langle \alpha_j^\vee, \alpha_j^\vee \rangle} = C_{ji}, \quad i, j \in \{1, \dots, n\}$$

Dual affine Weyl group

- the group generated by n reflections $r_{\alpha^\vee} = r_\alpha$, $\alpha^\vee \in \Delta^\vee$ coincides with W
- $\widehat{W}^{\mathrm{aff}}$ is generated by reflections $R^\vee = \{r_0^\vee, r_1, \dots, r_n\}$, where

$$r_0^\vee a = r_\eta a + \frac{2\eta}{\langle \eta,\, \eta\rangle}, \quad r_\eta a = a - \frac{2\langle a,\, \eta\rangle}{\langle \eta,\, \eta\rangle}\eta, \quad a \in \mathbb{R}^n$$

Dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$

$$\widehat{W}^{\mathrm{aff}} = Q \rtimes W$$

- the fundamental domain $F^\vee$ of $\widehat{W}^{\mathrm{aff}}$

$$F^\vee = \left\{0, \frac{\omega_1}{m_1^\vee}, \dots, \frac{\omega_n}{m_n^\vee}\right\}_\kappa$$

- a decomposition of $R^\vee = R^{s\vee} \cup R^{l\vee}$:

$$R^{s\vee} = \{r_\alpha \,|\, \alpha \in \Delta_s\} \cup \{r_0^\vee\}$$
$$R^{l\vee} = \{r_\alpha \,|\, \alpha \in \Delta_l\}$$

Dual fundamental domains

- two subsets of boundaries of $F^\vee$:

$$H^{s\vee} = \{a \in F^\vee \mid (\exists r \in R^{s\vee})(ra = a)\}$$
$$H^{l\vee} = \left\{a \in F^\vee \mid (\exists r \in R^{l\vee})(ra = a)\right\}$$

- fundamental domains $F^{s\vee} \subset F^\vee$, $F^{l\vee} \subset F^\vee$

$$F^{s\vee} = F^\vee \setminus H^{s\vee}$$
$$F^{l\vee} = F^\vee \setminus H^{l\vee}$$

- the symbols z_i^s, $z_i^l \in \mathbb{R}$, $i = 0, \dots, n$

$$z_i^s > 0, \quad z_i^l \geq 0, \quad r_i \in R^{s\vee}$$
$$z_i^s \geq 0, \quad z_i^l > 0, \quad r_i \in R^{l\vee}$$

$$F^{s\vee} = \left\{ z_1^s \omega_1 + \dots + z_n^s \omega_n \,\middle|\, z_0^s + z_1^s m_1^\vee + \dots + z_n^s m_n^\vee = 1 \right\}$$
$$F^{l\vee} = \left\{ z_1^l \omega_1 + \dots + z_n^l \omega_n \,\middle|\, z_0^l + z_1^l m_1^\vee + \dots + z_n^l m_n^\vee = 1 \right\}$$

Grids Λ^s_M and Λ^l_M

- $M \in \mathbb{N}$, W-invariant lattice P
- W-invariant finite group P/MQ
- number of elements of P/MQ is cM^n

The grid Λ_M

$$\Lambda_M \equiv MF^\vee \cap P/MQ$$

The grids $\Lambda^s_M \subset \Lambda_M$ and $\Lambda^l_M \subset \Lambda_M$

$$\Lambda^s_M \equiv MF^{s\vee} \cap P/MQ$$

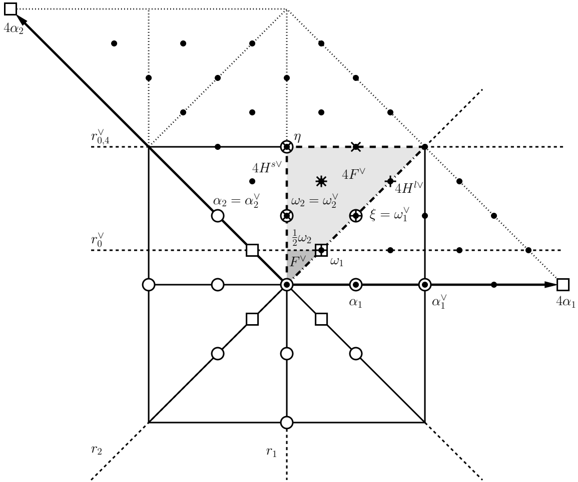
$$\Lambda^l_M \equiv MF^{l\vee} \cap P/MQ$$

Proposition

For the numbers of elements of the sets Λ^s_M and Λ^l_M it holds that

$$|\Lambda^s_M| = |F^s_M|,$$
$$|\Lambda^l_M| = |F^l_M|.$$

Grids $F_4^{s\vee}$ and $F_4^{l\vee}$ of C_2



Discrete orthogonality of S^s- and S^l-functions

- functions φ^s, φ^l on the grids F^s_M and F^l_M

$$\varphi^s_{b+MQ}(u) = \varphi^s_b(u),\ u \in F^s_M$$
$$\varphi^l_{b+MQ}(u) = \varphi^l_b(u),\ u \in F^l_M$$

- φ^s_λ, φ^l_λ with $\lambda \in P/MQ$, moreover $\lambda \in \Lambda_M$
- zero values:

$$\varphi^s_\lambda(u) = 0, \quad \lambda \in MH^{s\vee} \cap \Lambda_M,\, u \in F^s_M$$
$$\varphi^l_\lambda(u) = 0, \quad \lambda \in MH^{l\vee} \cap \Lambda_M,\, u \in F^l_M$$

Corollary

$$\varphi^s_\lambda(u),\, u \in F^s_M,\, \lambda \in \Lambda^s_M$$
$$\varphi^l_\lambda(u),\, u \in F^l_M,\, \lambda \in \Lambda^l_M$$

Outline

1. Short and long orbit functions
   - Lie groups/Lie algebras
   - Sign homomorphisms
   - S^s- and S^l-functions
2. **Discretization of orbit functions**
   - Grids F^s_M and F^l_M
   - Grids Λ^s_M and Λ^l_M
   - Discrete orthogonality of S^s- and S^l-functions

Discrete orthogonality of S^s- and S^l-functions

- $x \in \mathbb{R}^n/Q^\vee$, the isotropy group

$$\mathrm{Stab}(x) = \{w \in W \,|\, wx = x\}$$

- the orbit

$$Wx = \{wx \in \mathbb{R}^n/Q^\vee \,|\, w \in W\}$$

- $h_x \equiv |\mathrm{Stab}(x)|$, $\varepsilon(x) \equiv |Wx|$

$$\varepsilon(x) = \frac{|W|}{h_x}$$

- $\lambda \in \mathbb{R}^n/MQ$, the isotropy group

$$\mathrm{Stab}^\vee(\lambda) = \{w \in W \,|\, w\lambda = \lambda\}$$

- $h^\vee_\lambda \equiv |\mathrm{Stab}^\vee(\lambda)|$

Discrete orthogonality of S^s- and S^l-functions

- a scalar product for $f, g : F_M^s \to \mathbb{C}$

$$\langle f,\, g\rangle_{F_M^s} = \sum_{x\in F_M^s} \varepsilon(x) f(x)\overline{g(x)}$$

- a scalar product for $f, g : F_M^l \to \mathbb{C}$

$$\langle f,\, g\rangle_{F_M^l} = \sum_{x\in F_M^l} \varepsilon(x) f(x)\overline{g(x)}$$

Theorem

For $\lambda, \lambda' \in \Lambda_M^s$ it holds that

$$\langle \varphi_\lambda^s,\, \varphi_{\lambda'}^s\rangle_{F_M^s} = c\,|W|\,M^n h_\lambda^\vee \delta_{\lambda,\lambda'}$$

and for $\lambda, \lambda' \in \Lambda_M^l$ it holds that

$$\langle \varphi_\lambda^l,\, \varphi_{\lambda'}^l\rangle_{F_M^l} = c\,|W|\,M^n h_\lambda^\vee \delta_{\lambda,\lambda'}.$$

Discrete S^s- and S^l- transforms

- interpolating functions I^s_M, I^l_M

$$I^s_M(x) := \sum_{\lambda\in\Lambda^s_M} c^s_\lambda \varphi^s_\lambda(x), \qquad I^l_M(x) := \sum_{\lambda\in\Lambda^l_M} c^l_\lambda \varphi^l_\lambda(x), \quad x\in\mathbb{R}^n$$

- the interpolation of $f: F_M \to \mathbb{C}$: find c^s_λ (or c^l_λ)

$$I^s_M(x) = f(x), \quad x\in F^s_M$$
$$I^l_M(x) = f(x), \quad x\in F^l_M$$

Proposition

$$c^s_\lambda = \frac{\langle f, \varphi^s_\lambda\rangle_{F^s_M}}{\langle \varphi^s_\lambda, \varphi^s_\lambda\rangle_{F^s_M}} = (c\,|W|\,M^n h^\vee_\lambda)^{-1} \sum_{x\in F^s_M} \varepsilon(x) f(x) \overline{\varphi^s_\lambda(x)}$$

$$c^l_\lambda = \frac{\langle f, \varphi^l_\lambda\rangle_{F^l_M}}{\langle \varphi^l_\lambda, \varphi^l_\lambda\rangle_{F^l_M}} = (c\,|W|\,M^n h^\vee_\lambda)^{-1} \sum_{x\in F^l_M} \varepsilon(x) f(x) \overline{\varphi^l_\lambda(x)}$$

Discrete S^s- and S^l- transforms

Proposition (Plancherel formulas)

$$\sum_{x\in F^s_M} \varepsilon(x)\,|f(x)|^2 = c\,|W|\,M^n \sum_{\lambda\in\Lambda^s_M} h^\vee_\lambda |c^s_\lambda|^2$$

$$\sum_{x\in F^l_M} \varepsilon(x)\,|f(x)|^2 = c\,|W|\,M^n \sum_{\lambda\in\Lambda^l_M} h^\vee_\lambda |c^l_\lambda|^2.$$

- R. V. Moody, L. Motlochová, and J. Patera, *New families of Weyl group orbit functions*, arXiv:1202.4415
- J. Hrivnák, L. Motlochová, J. Patera, *On discretization of tori of compact simple Lie groups II,* J. Phys. A: Math. Theor. **45** (2012) 255201, arXiv:1206.0240
- J. Hrivnák, J. Patera, *On discretization of tori of compact simple Lie groups,* J. Phys. A: Math. Theor. **42** (2009) 385208
- R. V. Moody, J. Patera, *Orthogonality within the families of C-, S-, and E- functions of any compact semisimple Lie group*, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) 2 (2006) 076, 14 pages, math-ph/0611020
- A. Klimyk, J. Patera, *Antisymmetric orbit functions*, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) 3 (2007), paper 023, 83 pages; math-ph/0702040
- A. Klimyk, J. Patera, *E-orbit functions*, SIGMA (Symmetry,Integrability and Geometry: Methods and Applications) 4 (2008), 002, 57 pages; arXiv:0801.0822