Non-formal star-exponential of Kahlerian Lie groups

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Plan

Plan

1. Introduction to star-exponential
2. Star-products on negatively curved Kahlerian Lie groups
3. Star-exponential and applications

Plan

Formal construction on $\mathbb{R}^{2n}$

(Bayen Flato Fronsdal Lichnerowicz Sternheimer '78, Bayen Maillard '82)

Consider $(\mathbb{R}^{2n}, \omega_0)$ symplectic.

- Formal Moyal star-product: for $f, g \in C^\infty(\mathbb{R}^{2n})$,

$$f \star_\theta^0 g(x) = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{2^k k!} ((\omega_0)_{\mu\nu} \partial_\mu^x \partial_\nu^y)^k f(x) g(y)|_{y=x}$$

where $f \star_\theta^0 g \in C^\infty(\mathbb{R}^{2n})[[\theta]]$.

- Formal star-exponential:

$$\mathcal{E}_\theta(tf) = \sum_{k=0}^{\infty} \frac{(it)^k}{\theta^k k!} f^{(\star_\theta^0 k)}$$

lies in $C^\infty(\mathbb{R}^{2n})[[t, \theta]]$.

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(... Cahen Flato Gutt Sterheimer '85)

- For $f, g \in \mathcal{S}(\mathbb{R}^{2n})$, non-formal version of the Moyal product:

$$(f \star_\theta^0 g)(x) = \frac{1}{(\pi\theta)^{2n}} \int f(y) g(z) e^{-\frac{2i}{\theta}(\omega_0(x,y)+\omega_0(y,z)+\omega_0(z,x))} \mathrm{d}y\mathrm{d}z$$

- The formal star-exponential of $f \in \mathcal{S}(\mathbb{R}^{2n})$ converges to a distribution, which satisfies:

$$\partial_t \mathcal{E}_\theta(tf) = \frac{i}{\theta} f \star_\theta^0 \mathcal{E}_\theta(tf)$$

- Application: for $f \in \mathcal{S}(\mathbb{R}^{2n})$ and $g \in L^2(\mathbb{R}^{2n})$, define $T_f : g \mapsto f \star_\theta^0 g$.

$$\mathrm{Sp}(T_f) = \mathrm{supp}(\mathcal{F}_t(\mathcal{E}_\theta(tf)))$$

- Ex: if $f(x) = \lambda_y(x) = \omega_0(y, x)$, $\mathcal{E}_\theta(tf)(x) = e^{\frac{it}{\theta}\omega_0(y,x)}$.

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Construction on a Lie group

Let G be a connected simply connected Lie group acting on the symplectic manifold (M, ω) in a strong Hamiltonian way.

$$\lambda : \mathfrak{g} \to C^\infty(M)$$

- Given a star-product $\star_\theta$, we can define $\mathcal{E}_\theta(tf)$ as before.
- If $\star_\theta$ is covariant: $[\lambda_X, \lambda_Y]_{\star_\theta} = -i\theta\lambda_{[X,Y]}$, we have

$$\mathcal{E}_\theta(\lambda_X) \star_\theta \mathcal{E}_\theta(\lambda_Y) = \mathcal{E}_\theta(\lambda_{\mathrm{BCH}(X,Y)}) \qquad \text{(BCH)}$$

where $\mathrm{BCH}(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \dots$

$\to$ Representation $x \mapsto \mathcal{E}_\theta(\lambda_{\log(x)})$ of G.

(Nelson '59, Flato Simon Snellman Sternheimer '72)

- If M: coadjoint orbit of nilpotent G, then $(M, \omega) \approx (\mathbb{R}^{2n}, \omega_0)$ and $\star_\theta^0$ covariant. $\mathcal{E}_\theta(\lambda_X)(x) = e^{\frac{i}{\theta}P(X,x)}$. (Arnal Cortet '85, Arnal Gutt '87)

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Geometry of negatively curved Kahlerian Lie groups

(Pyatetskii-Shapiro '69)

- Elementary normal j-groups $\mathbb{S}$: AN part of $SU(1,n)$
- Lie algebra $\mathfrak{s} = \mathbb{R}H \oplus V \oplus \mathbb{R}E$ with

$$[H,x] = x, \quad [H,E] = 2E, \quad [x,x'] = \omega_0(x,x')E$$

- Coadjoint orbit M equivariantly diffeomorphic to $\mathbb{S}$.
- Ex: connected affine group ($V = 0$)

$$(a,\ell)(a',\ell') = (a+a', e^{-2a'}\ell + \ell')$$

The moment maps are $\lambda_H = 2\ell$ and $\lambda_E = e^{-2a}$.

- In general (normal j-groups),

$$G = (..(\mathbb{S}_N \ltimes \mathbb{S}_{N-1}) \ltimes \ldots \ltimes \mathbb{S}_2) \ltimes \mathbb{S}_1$$

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Schwartz space

(Bieliavsky Gayral '11)

- Adapted Schwartz space:

$$\mathcal{S}(G) = \{f \in C^\infty(G),\ \forall X \in \mathfrak{g},\ \forall Y \in \mathcal{U}(\mathfrak{g}),\ \forall k,\\ \sup |(\lambda_X)^k \tilde{Y} f| < \infty\}$$

- For the affine group $\mathbb{S}$:

$$\lambda_H = 2\ell, \quad \lambda_E = e^{-2a}, \quad \tilde{H} = \partial_a - 2\ell\partial_\ell, \quad \tilde{E} = \partial_\ell.$$

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Non-formal star-product

(Bieliavsky Massar '01, Bieliavsky Gayral '11)

- Since $\mathbb{S} \approx \mathbb{R}^2$, we can consider the non-formal Moyal product $\star_\theta^0$ on $\mathbb{S}$. It is $\mathbb{S}$-covariant.
- There exists operators T_θ on functions on $\mathbb{S}$ such that

$$\star_\theta = T_\theta(T_\theta^{-1}(\cdot) \star_\theta^0 T_\theta^{-1}(\cdot))$$

is $\mathbb{S}$-left-invariant: $\forall x \in \mathbb{S}$, $L_x^*(f \star_\theta g) = (L_x^* f) \star_\theta (L_x^* g)$.

- $(\mathcal{S}(\mathbb{S}), \star_\theta)$ is an associative algebra.

$$(f \star_\theta g)(a, \ell) = \int K(a, a_1, a_2) f(a_1, \ell_1) g(a_2, \ell_2)$$
$$e^{\frac{2i}{\theta}(\sinh(2(a_1 - a_2))\ell + \sinh(2(a_2 - a))\ell_1 + \sinh(2(a - a_1))\ell_2)} \mathrm{d}a_i \mathrm{d}\ell_i$$

- Extension of $\star_\theta$ by oscillatory integral.

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- Extension of $\star_\theta$ by oscillatory integral.

Formal star-exponential

- If $X = \alpha H + \beta E \in \mathfrak{s}$, equation for Moyal:

$$\partial_t \mathcal{E}^0_\theta(t\lambda_X) = \frac{i}{\theta}(2\alpha\ell + \beta e^{-2a}) \star^0_\theta \mathcal{E}^0_\theta(t\lambda_X)$$

- Solution, which satisfies BCH:

$$\mathcal{E}^0_\theta(t\lambda_X)(a,\ell) = e^{\frac{i}{\theta}(2\alpha t\ell + \frac{\beta}{\alpha}\sinh(\alpha t)e^{-2a})}$$

- For the invariant star-product: $\mathcal{E}_\theta(t\lambda_X) = T_\theta \mathcal{E}^0_\theta(t\lambda_X)$

Theorem (Bieliavsky A.G. Spinnler '12)

The formal star-exponential associated to $\star_\theta$ can be expressed as

$$\mathcal{E}_\theta(t\lambda_X)(a,\ell) = \cosh(\alpha t) e^{\frac{i}{\theta}\sinh(\alpha t)(2\ell + \frac{\beta}{\alpha}e^{-2a})}$$

in $C^\infty(\mathbb{S})[[t,\theta]]$ and satisfies the BCH formula.

- Generalization: elementary, semidirect products...

Formal star-exponential

- If $X = \alpha H + \beta E \in \mathfrak{s}$, equation for Moyal:

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Theorem (Bieliavsky A.G. Spinnler '12)

The expression of $\mathcal{E}_\theta(t\lambda_X)$ lies in $\mathcal{M}_{\star_\theta}(\mathcal{S}(\mathbb{S}))$. For $\theta \in \mathbb{R}^*$, the power series in θ of $\mathcal{E}_\theta(t\lambda_X)$ converges to this expression, which is therefore the (unique) non-formal star-exponential. It satisfies the BCH formula.

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- Expression of the formal star-exponential $\mathcal{E}_\theta(t\lambda_X)$
- Non-formal meaning: $\mathcal{E}_\theta(t\lambda_X)$ is in $\mathcal{M}_{\star_\theta}(\mathcal{S}(\mathbb{S}))$
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Consequences and perspectives:

- Access to the spectrum of multiplication operators
- New functional transformation: adapted to PDE?
- Representation theory of these groups

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