The leafwise Laplacian and its spectrum: the singular case

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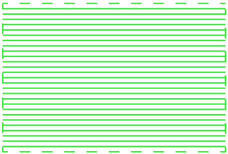
Summary

1.1 Definition: Foliation

Partition to connected submanifolds. Local picture: 

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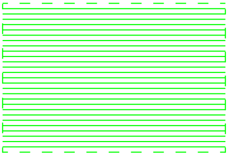
Partition to connected submanifolds. Local picture: 

In other words: There is an open cover of M by foliation charts of the form $\Omega = U \times T$, where $U \subseteq \mathbb{R}^p$ and $T \subseteq \mathbb{R}^q$.

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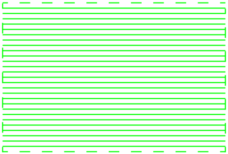
Partition to connected submanifolds. Local picture: 

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T is the transverse direction and U is the longitudinal or leafwise direction.

The change of charts is of the form $f(u, t) = (g(u, t), h(t))$.

1.1 Laplacians

Each leaf is a complete Riemannian manifold:

Laplacian Δ_L acting on $L^2(L)$

The family of leafwise Laplacians:

Laplacian Δ_M acting on $L^2(M)$

Statement of 3 theorems

Theorem 1 (Connes, Kordyukov)

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- more generally for every leafwise elliptic (pseudo-)differential operator.

Not trivial because:

- Δ_M not elliptic (as an operator on M).
- L not compact.

Spectrum of the Laplacian

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Theorem 3 (Connes)

In many cases, one can predict the possible gaps in the spectrum.

The same is true for all leafwise elliptic operators.

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- Case of a single leaf:

  Take any $(x, y) \in M \times M \rightsquigarrow C^*(M, F) = \mathcal{K}(L^2(M))$

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- a product, a fibre bundle $p : M \to B$:

  Take $(x,y) \in M \times M$ s.t. $p(x) = p(y) \rightsquigarrow C^*(M,F) = C(B) \otimes \mathcal{K}$

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- General case: $(x, y) \in M \times M$ s.t. x, y in same leaf L;
  - γ path on L connecting x, y; h_γ path holonomy depends only on homotopy class of γ
  - $H(F) = \{(x, \mathrm{germ}(h_\gamma), y)\}$ Holonomy groupoid.
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$C^*(M, F) =$ continuous functions on "space of leaves M/F".

2.2 Pseudodifferential operators (Connes)

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Using Fourier transform one can write a differential operator P (acting by left multiplication on $f \in C_c^\infty(G)$) as:

$$(Pf)(x, y) = \int exp(i\langle \phi(x, z), \xi\rangle)\alpha(x, \xi)\chi(x, z)f(z, y)d\xi dz$$

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Where

- ϕ is the phase: through a local diffeomorphism defined on an open subset $\widetilde{\Omega} \simeq U \times U \times T \subset G$ (where $\Omega = U \times T$ is a foliation chart). $\phi(x,z) = x - z \in F_x$;
- χ is the cut-off function: χ smooth, $\chi(x,x) = 1$ on (a compact subset of) Ω, $\chi(x,z) = 0$ for $(x,z) \notin \widetilde{\Omega}$;
- $\alpha \in C^\infty(F^*)$ is a polynomial on ξ. It is called the symbol of P.

More general symbols

We can make sense of an expression like that for much more general symbols, in particular poly-homogeneous ones:

$$\alpha(u, \xi) \sim \sum_{k \in \mathbb{N}} \alpha_{m-k}(u, \xi)$$

where α_j homogeneous of degree j (outside a neighborhood of $M \subset F^*$).

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Proposition (Connes)

- Negative order pseudodifferential operators $\in C^*(M,F)$
- Zero order pseudodifferential operators: multipliers of $C^*(M,F)$.

Longitudinal pseudodifferential calculus

Together with multiplicativity of the principal symbol this gives an exact sequence of C^*-algebras:

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Theorem (Connes, Kordyukov, Vassout)

Elliptic operators of positive order are regular unbounded multipliers (in the sense of Baaj-Woronowicz: $graph(D) \oplus graph(D)^{\perp}$ is dense).

2.3 Proof of theorems 1 and 2

$L^2(M)$ and $L^2(L)$: representations of the foliation C^*-algebras.

Proposition (Baaj, Woronowicz)

Every representation extends to regular multipliers.

image of the adjoint = adjoint of the image

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Proposition

Every injective morphism of C^*-algebras is isometric and isospectral.

Proposition (Fack-Skandalis)

If the foliation is minimal (*i.e.* all leaves are dense) then the foliation C^*-algebra is simple.

Theorem 2 follows.

Examples for Theorem 3 (Connes)

Horocyclic foliation: no gaps in the spectrum

Let the "$ax+b$" group act on a compact manifold M.
e.g. $M = SL(2,\mathbb{R})/\Gamma$ where Γ discrete co-compact group.
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- The "ax" subgroup $\longrightarrow$ action of $\mathbb{R}^*_+$ which scales the trace.
- Image of K_0 countable subgroup of $\mathbb{R}$, invariant under $\mathbb{R}^*_+$ action.

Examples for Theorem 3

Application:

- $M = SL(2,\mathbb{R})/\Gamma$ as before; injection $\iota : \mathbb{R} \to M$

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Similarly, Kronecker flow: Image of the trace $\mathbb{Z} + \theta\mathbb{Z}$

Can be a Cantor type set

Frobenius...

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Δ only depends on the bundle $F \subset TM$ of vector fields tangent to the leaf.

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Conversely

Frobenius Theorem

Every integrable subbundle of the tangent bundle corresponds to a foliation.

3.1 Almost injective algebroids

Serre-Swan Theorem

Bundles = Finitely generated projective $C^\infty(M)$-modules.

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Equivalently:

Lie algebroid with anchor $A_x \to T_x M$, injective in a dense set.
Image F_x. Dimension lower semi-continuous.

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Theorem (Debord, Pradines, Bigonnet)

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Baum-Connes predicts the K-theory and is known to hold in many cases...

3.2 Stefan-Sussmann foliations

Definition (Stefan, Sussmann, A-Skandalis)

A (singular) foliation is a finitely generated sub-module $\mathcal{F}$ of $C^\infty(M; TM)$, stable under brackets.

No longer projective. The fiber $\mathcal{F}/I_x\mathcal{F}$: upper semi-continuous dimension.

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2. $\mathbb{R}^2$ foliated by 2 leaves: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$.
   No obvious best choice. $\mathcal{F}$ given by the action of a Lie group

$$GL(2,\mathbb{R}), SL(2,\mathbb{R}), \mathbb{C}^*$$

Constructions of A-Skandalis

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And also

- Analytic index (element of $KK(C_0(\mathcal{F}^*); C^*(M, \mathcal{F}))$)
- tangent groupoid + defines same KK element.

Holonomy transformations I: Regular case

$\mathcal{F}$ sections of F involutive subbundle of TM.

$\gamma : [0,1] \to M$ path on a leaf, S_x, S_y transversals at $x = \gamma(0), y = \gamma(1)$

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For any t, extend $\frac{d}{ds}|_{s=t}\, \gamma(s)$ to a time-dependent v.f $Z_t \in \mathcal{F}$

Define $\Gamma : S_x \times [0,1] \to M$ following the flow of Z_t on points of S_x.
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Does not depend on choice of Z_t. Get maps

- $\{$homotopy classes of paths $\gamma\} \mapsto \mathrm{GermAut}_{\mathcal{F}}(S_x; S_y)$ (holonomy)
- $\{$homotopy classes of paths $\gamma\} \mapsto \mathrm{Iso}(T_x S_x; T_y S_y)$ (linear holonomy)

Holonomy transformations II: Singular case

Take $M = \mathbb{R}$, $\mathcal{F} = \langle x\frac{\partial}{\partial x}\rangle$ and $x = y = 0$.

Transversal = neighborhood of 0 in $\mathbb{R}$.

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Constant path $\gamma(t) = 0$ admits many extensions, e.g.

1. flow of zero vector field: $\Gamma : S_0 \times [0,1] \to S_0, \quad (x,t) \mapsto x$;
2. flow of $x\frac{\partial}{\partial x}$: $\Gamma(x,t) = e^t x$

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Observation 1 (A-Zambon)

Different choices of Γ differ by the flow of $X \in \mathcal{F}(x) = \{X \in \mathcal{F} : X(x) = 0\}$.

Hence $\Gamma(\cdot, 1)$ gives a class in $\frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_x)}{\exp(\mathcal{F}(x))}$

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Observation 2 (A-Zambon)

Not linearizable! To make it linearizable, must consider $\frac{\mathrm{GermAut}_{\mathcal{F}}(S_x,S_x)}{\exp(I_x\mathcal{F})}$.

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- Find $U \subset \mathbb{R}^n \times M$ neighborhood of $(x, 0)$ where $t : U \to M$ is defined:

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A-Skandalis

Using bi-submersions can construct $C^*(\mathcal{F})$ and longitudinal pseudodifferential calculus!

Generalization of Theorem 1

Theorem 1 (A-Skandalis)

Let M be a smooth compact manifold. Let $X_1, \ldots, X_N \in C^\infty(M; TM)$ be smooth vector fields such that $[X_i, X_j] = \sum_{k=1}^N f_{ij}^k X_k$.

Then $\Delta = \sum X_i^* X_i$ is essentially self-adjoint (both in $L^2(M)$ and $L^2(L)$).

Proof

This operator is indeed a regular unbounded multiplier of our C^*-algebra.

Generalization of Theorem 2

Theorem (Skandalis)

Assume that the (dense open) set $\Omega \subset M$ where leaves have maximal dimension is Lebesgue measure 1. Assume that the restriction of $\mathcal{F}$ to Ω is minimal and that the holonomy groupoid of this restriction is Hausdorff and amenable.

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The C^*-algebra $C^*(\Omega, \mathcal{F})$ is simple (Fack-Skandalis) and sits as a two-sided ideal in $C^*(M, \mathcal{F})$. The natural representations of $C^*(M, \mathcal{F})$ to $L^2(L)$ and $L^2(M)$ are extensions to multipliers of faithful representations of $C^*(\Omega, \mathcal{F})$. They are weakly equivalent.

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The singular extension of the foliation to the closure M of Ω is used to prove Δ_M is regular. Furthermore, Δ_M depends on the way $\mathcal{F}$ is extended.

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Need to know the "shape" of $K_0(C^*(M, \mathcal{F}))$.

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1. A - M. Zambon: Longitudinal smoothness controlled by "essential isotropy groups" attached to each leaf. When discrete, groupoid longitudinally smooth.

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Answers...

1. A - M. Zambon: Longitudinal smoothness controlled by "essential isotropy groups" attached to each leaf. When discrete, groupoid longitudinally smooth.
2. Conjecture: Baum-Connes true for $\mathcal{F}$ iff true for each leaf.

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