The leafwise Laplacian and its spectrum: the singular case

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Summary

1. Introduction
   - Foliations and Laplacians
   - Statement of 3 theorems

2. How to prove these theorems
   - The $C^*$-algebra of a foliation
   - Pseudodifferential calculus
   - Proofs

3. The singular case
   - Almost regular foliations
   - Stefan-Sussmann foliations

4. Generalizations: Singular foliations
1.1 Definition: Foliation

Partition to connected submanifolds. Local picture:
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In other words: There is an open cover of $\mathcal{M}$ by foliation charts of the form $\Omega = U \times T$, where $U \subseteq \mathbb{R}^p$ and $T \subseteq \mathbb{R}^q$.

$T$ is the transverse direction and $U$ is the longitudinal or leafwise direction.
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The change of charts is of the form $f(u, t) = (g(u, t), h(t))$. 

1.1 Laplacians

Each leaf is a complete Riemannian manifold:

\[ \text{Laplacian } \Delta_L \text{ acting on } L^2(L) \]

The family of leafwise Laplacians:

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Statement of 3 theorems

Theorem 1 (Connes, Kordyukov)

$\Delta_M$ and $\Delta_L$ are essentially self-adjoint.
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Also true (and more interesting)

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- more generally for every leafwise elliptic (pseudo-)differential operator.
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Not trivial because:

- \( \Delta_M \) not elliptic (as an operator on \( M \)).
- \( L \) not compact.
Spectrum of the Laplacian

**Theorem 2 (Kordyukov)**

If $L$ is dense + amenability assumptions, $\Delta_M$ and $\Delta_L$ have the same spectrum.
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Theorem 3 (Connes)

In many cases, one can predict the possible gaps in the spectrum.

The same is true for all leafwise elliptic operators.
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Its construction: Completion of a convolution algebra
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Kernels $k(x, y)$: $k_1 \ast k_2 = \int k_1(x, z)k_2(z, y)dz$

- Case of a single leaf:
  Take any $(x, y) \in M \times M \leadsto C^*(M, F) = \mathcal{K}(L^2(M))$
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  Take any $(x, y) \in M \times M \sim C^*(M, F) = \mathcal{K}(L^2(M))$

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  Take $(x, y) \in M \times M$ s.t. $p(x) = p(y) \sim C^*(M, F) = C(B) \otimes \mathcal{K}$
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- General case: $(x, y) \in M \times M$ s.t. $x, y$ in same leaf $L$;
  
  - $\gamma$ path on $L$ connecting $x, y$; $h_\gamma$ path holonomy depends only on homotopy class of $\gamma$
  
  - $H(F) = \{(x, \text{germ}(h_\gamma), y)\}$ Holonomy groupoid.
  
  - topology, manifold structure $\Rightarrow H(F)$ is a Lie groupoid (not always Hausdorff).
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$\mathbb{C}^*(M, F) = \text{continuous functions on "space of leaves } M/F"$. 
2.2 Pseudodifferential operators (Connes)

The Lie algebra of vector fields tangent to the foliation acts by unbounded multipliers on $C_c^\infty(G)$. The algebra generated is the algebra of differential operators.

Using Fourier transform one can write a differential operator $P$ (acting by left multiplication on $f \in C_c^\infty(G)$) as:

$$ (Pf)(x, y) = \int \exp(i\langle \phi(x, z), \xi \rangle) \alpha(x, \xi) \chi(x, z) f(z, y) d\xi dz $$

Where $\phi$ is the phase: through a local diffeomorphism defined on an open subset $\tilde{\Omega} \cong U \times U \times T \subset G$ (where $\Omega = U \times T$ is a foliation chart).

$\phi(x, z) = x - z \in F_x$; $\chi$ is the cut-off function: $\chi \text{ smooth, } \chi(x, x) = 1 \text{ on (a compact subset of) } \Omega, \chi(x, z) = 0 \text{ for } (x, z) \notin \tilde{\Omega}$;

$\alpha \in C_c^\infty(F_x^*)$ is a polynomial on $\xi$. It is called the symbol of $P$. 

I. Androulidakis (Athens)
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More general symbols

We can make sense of an expression like that for much more general symbols, in particular \textbf{poly-homogeneous} ones:

$$\alpha(u, \xi) \sim \sum_{k \in \mathbb{N}} \alpha_{m-k}(u, \xi)$$

where $\alpha_j$ homogeneous of degree $j$ (outside a neighborhood of $M \subset \mathbb{F}^*$).

- $m$ is called the \textbf{order} of $\alpha$ and the associated operator;
- $\alpha_m$ is the \textbf{principal symbol}.
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**Proposition (Connes)**

- Negative order pseudodifferential operators $\in C^*(M, F)$
- Zero order pseudodifferential operators: **multipliers** of $C^*(M, F)$. 
Longitudinal pseudodifferential calculus

Together with multiplicativity of the principal symbol this gives an exact sequence of $C^*$-algebras:

$$0 \to C^*(\mathcal{M}, F) \to \Psi^*(\mathcal{M}, F) \to C(SF^*) \to 0$$
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**Theorem (Connes, Kordyukov, Vassout)**

Elliptic operators of positive order are **regular** unbounded multipliers (in the sense of Baaj-Woronowicz: $\text{graph}(D) \oplus \text{graph}(D)^\perp$ is dense).
2.3 Proof of theorems 1 and 2

$L^2(M)$ and $L^2(L)$: representations of the foliation $C^*$-algebras.

Proposition (Baaj, Woronowicz)

Every representation extends to regular multipliers.

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\text{image of the adjoint} = \text{adjoint of the image}
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Every injective morphism of $C^*$-algebras is isometric and isospectral.

**Proposition (Fack-Skandalis)**

If the foliation is minimal (i.e. all leaves are dense) then the foliation $C^*$-algebra is simple.

Theorem 2 follows.
Examples for Theorem 3 (Connes)

**Horocyclic foliation: no gaps in the spectrum**

Let the "$ax + b$" group act on a compact manifold $M$.

e.g. $M = \text{SL}(2, \mathbb{R})/\Gamma$ where $\Gamma$ discrete co-compact group.

Leaves = orbits of the "$x + b$" group (assume it is minimal).

The spectrum of the Laplacian is an interval $[m, +\infty)$.
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- gaps in the spectrum \(\rightarrow\) projections in \( C^* (M, F) \).
- \( \exists \) invariant measure by \( ax + b \) \(\rightarrow\) trace on \( C^* (M, F) \) faithful since \( C^* (M, F) \) simple (Fack-Skandalis).
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- The "ax" subgroup \( \rightarrow \) action of \( \mathbb{R}^*_+ \) which scales the trace.
- Image of \( K_0 \) countable subgroup of \( \mathbb{R} \), invariant under \( \mathbb{R}^*_+ \) action.
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Application:

- $M = \text{SL}(2, \mathbb{R})/\Gamma$ as before; injection $\iota : \mathbb{R} \to M$
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- whence: connected spectrum of operators on $L^2(\mathbb{R})$ of the form

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where $V = f \circ i$, for $f$: continuous (positive) function.
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Similarly, Kronecker flow: Image of the trace $\mathbb{Z} + \theta \mathbb{Z}$

Can be a Cantor type set
Frobenius…

Remark

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Vectors tangent to the leaves: Subbundle \( F \) of the tangent bundle.

It is an **integrable subbundle**: If \( X \) and \( Y \) are vector fields tangent to \( F \) then Lie bracket \( [X, Y] \) is tangent to \( F \).
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Conversely

Frobenius Theorem
Every integrable subbundle of the tangent bundle corresponds to a foliation.
3.1 Almost injective algebroids

**Serre-Swan Theorem**

Bundles = Finitely generated projective $C^\infty(M)$-modules.

\[
\begin{array}{c}
E & \leftrightarrow & C^\infty(M; E) \\
\end{array}
\]

Debord's setting

$A$: finitely generated projective sub-module of $C^\infty(M)$, stable under brackets.

Equivalently: Lie algebroid with anchor $A_x \rightarrow T_x M$, injective in a dense set.

Image $F_x$. Dimension lower semi-continuous.
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In other words, it is the Lie algebroid of a Lie groupoid, whence

- $C^*$-algebra (**Renault**)
- pseudodifferential calculus (**Connes, Monthubert-Pierrot, Nistor-Weinstein-Xu**)
- Elliptic operators: regular multipliers (**Vassout**)

Furthermore, well-defined Laplacian

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Theorem 3: No gaps for a manifold with conic singularities obtained using a finite covolume subgroup of $SL_2(\mathbb{R})$

Baum-Connes predicts the $K$-theory and is known to hold in many cases...
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3.2 Stefan-Sussmann foliations

**Definition (Stefan, Sussmann, A-Skandalis)**

A (singular) foliation is a finitely generated sub-module $\mathcal{F}$ of $C^\infty(M; TM)$, stable under brackets.

No longer projective. The fiber $\mathcal{F}/I_x\mathcal{F}$: upper semi-continuous dimension. One may still define leaves (Stefan-Sussmann).

Examples

1. $\mathbb{R}$ foliated by 3 leaves: $(-\infty, 0)$, $\{0\}$, $(0, +\infty)$. $\mathcal{F}$ generated by $x^n \partial/\partial x$. Different foliation for every $n$.

2. $\mathbb{R}^2$ foliated by 2 leaves: $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. No obvious best choice. $\mathcal{F}$ given by the action of a Lie group $\text{GL}(2, \mathbb{R}), \text{SL}(2, \mathbb{R}), C^*$. Androulidakis (Athens)
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And also

- Analytic index (element of $\text{KK}(\mathbb{C}_0(\mathcal{F}^*); \mathbb{C}^*(M, \mathcal{F}))$)
- tangent groupoid $+$ defines same KK element.
Holonomy transformations I: Regular case

$\mathcal{F}$ sections of $\mathcal{F}$ involutive subbundle of $TM$.

$\gamma : [0, 1] \to M$ path on a leaf, $S_x, S_y$ transversals at $x = \gamma(0), y = \gamma(1)$
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For any $t$, extend $\frac{d}{ds} |_{s=t} \gamma(s)$ to a time-dependent v.f $Z_t \in \mathcal{F}$

Define $\Gamma: S_x \times [0, 1] \to M$ following the flow of $Z_t$ on points of $S_x$.
(Assume $\Gamma(q, 1) \subseteq S_y$).
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Define holonomy of $\gamma$ the germ at $x$ of

$$\text{hol}_\gamma : S_x \to S_y \quad q \mapsto \Gamma(q, 1)$$
Holonomy transformations I: Regular case

\( \mathcal{F} \) sections of \( F \) involutive subbundle of \( \mathcal{T}M \).

\( \gamma : [0, 1] \to M \) path on a leaf, \( S_x, S_y \) transversals at \( x = \gamma(0), y = \gamma(1) \)

For any \( t \), extend \( \frac{d}{ds} \big|_{s=t} \gamma(s) \) to a time-dependent v.f \( Z_t \in \mathcal{F} \)

Define \( \Gamma : S_x \times [0, 1] \to M \) following the flow of \( Z_t \) on points of \( S_x \).
( Assume \( \Gamma(q, 1) \subseteq S_y \).)

Define \textit{holonomy of} \( \gamma \) the germ at \( x \) of

\[ \text{hol}_\gamma : S_x \to S_y \quad q \mapsto \Gamma(q, 1) \]

Does not depend on choice of \( Z_t \). Get maps

- \{homotopy classes of paths \( \gamma \} \mapsto \text{GermAut}_\mathcal{F}(S_x; S_y) \) (holonomy)
- \{homotopy classes of paths \( \gamma \} \mapsto \text{Iso}(T_xS_x; T_yS_y) \) (linear holonomy)
Holonomy transformations II: Singular case

Take $M = \mathbb{R}$, $\mathcal{F} = \langle x \frac{\partial}{\partial x} \rangle$ and $x = y = 0$.

Transversal = neighborhood of 0 in $\mathbb{R}$. 

Observation 1 (A-Zambon)
Different choices of $\Gamma$ differ by the flow of $X \in \mathcal{F}(x) = \{X \in \mathcal{F}: X(x) = 0\}$.

Hence $\Gamma(\cdot, 1)$ gives a class in $\text{GermAut}_x \mathcal{F}(\mathbb{R}_x, \mathbb{R}_x) \exp(\mathcal{F}(x))$.

Observation 2 (A-Zambon)
Not linearizable! To make it linearizable, must consider $\text{GermAut}_x \mathcal{F}(\mathbb{R}_x, \mathbb{R}_x) \exp(\mathcal{I}_x \mathcal{F})$. 

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1. flow of zero vector field: $\Gamma : S_0 \times [0, 1] \to S_0$, $(x, t) \mapsto x$;
2. flow of $x \frac{\partial}{\partial x}$: $\Gamma(x, t) = e^t x$
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- Take $X_1, \ldots, X_n \in \mathcal{F}$ generating $\mathcal{F}$.
- Find $U \subset \mathbb{R}^n \times M$ neighborhood of $(x, 0)$ where $t : U \rightarrow M$ is defined:

$$t(\lambda_1, \ldots, \lambda_n, y) = \exp_y \left( \sum_{i=1}^{n} \lambda_i X_i \right)$$
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- Put \( s = \text{pr}_2 \). Then \( s, t : U \rightarrow \mathcal{M} \) submersions and \( U \) foliated by

\[
    s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = C^\infty(U; \ker ds) + C^\infty(U; \ker dt)
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Leaves: \( s^{-1}(L) \cap t^{-1}(L) \) where \( L \) leaf of \( \mathcal{F} \).
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A bisection $b$ of $s, t$ carries a holonomy $h \in \text{Aut}_\mathcal{F}(M)$. 

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Whence $\# : U \to H(\mathcal{F})$ is a smooth cover of an open subset of $H(\mathcal{F})$. 

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Generalization of Theorem 1

**Theorem 1 (A-Skandalis)**

Let $M$ be a smooth compact manifold. Let $X_1, \ldots, X_N \in C^\infty(M; TM)$ be smooth vector fields such that $[X_i, X_j] = \sum_{k=1}^{N} f_{ij}^k X_k$.

Then $\Delta = \sum X_i^* X_i$ is essentially self-adjoint (both in $L^2(M)$ and $L^2(L)$).

**Proof**

This operator is indeed a regular unbounded multiplier of our $C^*$-algebra.
Generalization of Theorem 2

**Theorem (Skandalis)**

Assume that the (dense open) set $\Omega \subset M$ where leaves have maximal dimension is Lebesgue measure 1. Assume that the restriction of $\mathcal{F}$ to $\Omega$ is minimal and that the holonomy groupoid of this restriction is Hausdorff and amenable.

Then $\Delta_M$ and $\Delta_L$ have the same spectrum (leaf $L \subset \Omega$).

Proof

The $C^*$-algebra $C^*(\Omega, \mathcal{F})$ is simple (Fack-Skandalis) and sits as a two-sided ideal in $C^*(M, \mathcal{F})$. The natural representations of $C^*(M, \mathcal{F})$ to $L^2(L)$ and $L^2(M)$ are extensions to multipliers of faithful representations of $C^*(\Omega, \mathcal{F})$. They are weakly equivalent.

The singular extension of the foliation to the closure $M$ of $\Omega$ is used to prove $\Delta_M$ is regular. Furthermore, $\Delta_M$ depends on the way $\mathcal{F}$ is extended.
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**Answers...**

1. A - M. Zambon: Longitudinal smoothness controlled by ”essential isotropy groups” attached to each leaf. When discrete, groupoid longitudinally smooth.
2. Conjecture: Baum-Connes true for $F$ iff true for each leaf.
Papers


