Invitation to Hilbert C^*-modules and Morita-Rieffel equivalence

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1) Introduction

(**1953**) I. Kaplansky, *Modules Over Operator Algebras*, Amer. J. Math.

> *"... extension of the theory of modules to over non-commutative C^*-algebras presents many difficulties"*

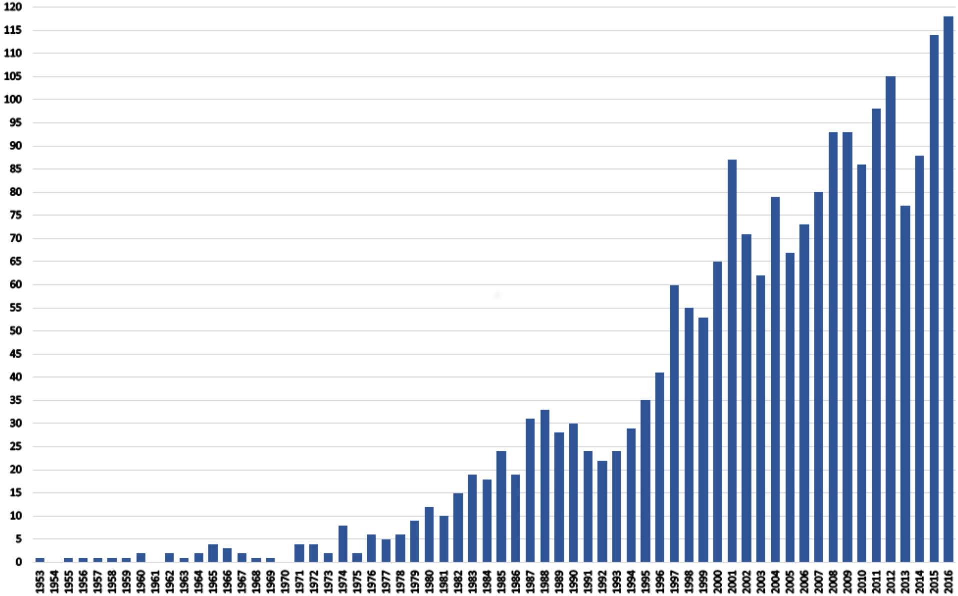
This claim was disproved by:

(**1973**) W. L. Paschke, *Inner product modules over B^*-algebras*, Trans. AMS.

Main areas of applications of Hilbert C^*-modules:

- Induced representations and Morita equivalence - Rieffel (1974) ...
- KK-theory - Kasparov (1975) ...
- C^*-algebraic quantum groups - Woronowicz (1991) ...
- Universal C^*-algebras - Pimsner (1998) ...

Publications wherein Hilbert C^*-modules play a fundamental role:



2) Hilbert C^*-modules

Def.

A **pre-Hilbert space** is a complex linear space X equipped with a map $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ such that, for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{C}$:

(1) $\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$

(2) $\overline{\langle x, y \rangle} = \langle y, x \rangle$

(3) $\langle x, x \rangle \geq 0$

(4) $\langle x, x \rangle = 0 \implies x = 0$

The map $\langle x, y \rangle$ is called an **inner-product**. It follows (exercise) that

$$\|x\| := \sqrt{|\langle x, x \rangle|}$$

is a norm on X. We say that X is a **Hilbert space** if it is complete with respect to the norm defined above.

Rem.

X is a complex linear space $\equiv$ X is a $\mathbb{C}$-module

complex numbers $\mathbb{C} \equiv$ one dimensional C^*-algebra

Def. Let A be a C^*-algebra.

A (right) **pre-Hilbert** A**-module** is a (right) A-module X equipped with a map $\langle \cdot, \cdot \rangle_A : X \times X \to A$ such that, for all $x, y, z \in X$ and $a, b \in A$:

(1) $\langle x, ya + zb \rangle_A = \langle x, y \rangle_A \, a + \langle x, z \rangle_A \, b$

(2) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$

(3) $\langle x, x \rangle_A \geq 0$ (positivity in A)

(4) $\langle x, x \rangle_A = 0 \implies x = 0$

The map $\langle x, y \rangle_A$ is called an A**-valued inner-product**. It follows (exercise) that

$$\|x\| := \sqrt{\|\langle x, x \rangle_A\|}$$

is a "norm" on X. We say that X is a (right) **Hilbert** A**-module** if it is complete with respect to the metric $d(x, y) = \|x - y\|$.

Rem. If A has the unit 1 (in general we may use approximate units), putting

$$\lambda x := x(\lambda 1), \qquad \lambda \in \mathbb{C}, \ x \in X$$

X becomes a complex linear space (a complex Banach space).

Ex.1 (Hilbert spaces) $\{$Hilbert $\mathbb{C}$-modules$\} = \{$Hilbert spaces$\}$.

Ex.2 (C^*-algebras) Let A be a C^*-algebra. The linear space $A_A := A$ with operations (where $x, y \in A_A$, $a \in A$):

$$x \cdot a := xa, \qquad \langle x, y \rangle_A := x^* y,$$

is a Hilbert A-module. Thus $\{C^*$-algebras$\} \subseteq \{$Hilbert C^*-modules$\}$!!!

Closed right ideals J in A correspond to Hilbert A-submodules J_A of A_A.

Ex.3 (Concrete Hilbert A-modules) Let A be a C^*-subalgebra of $B(H)$ where H is a Hilbert space H. Let $X \subseteq B(H)$ be closed subspace such that

$$XA \subseteq X \quad \text{and} \quad X^* X \subseteq A.$$

Then X with operations inherited from $B(H)$ is a Hilbert A-module.

Every Hilbert A-module can be represented in this form!!! (Murphy 1997)

Ex.4 (Hilbert $C(M)$-modules "=" Vector bundles) Let $H = (\{H_t\}_{t\in M}, \Gamma(H))$ be a continuous field of Hilbert spaces over a compact Hausdorff space M, i.e.:

(1) $\{H_t\}_{t\in M}$ is a family of Hilbert spaces,

(2) $\Gamma(H)$ is a linear subspace of sections $M \ni t \mapsto x(t) \in H_t$ such that $M \ni t \mapsto \|x(t)\|$ is continuous,

(3) $H_t = \overline{\{x(t) : x \in \Gamma(H)\}}$ for each $t \in M$,

(4) If x is a section and for every $t_0 \in M$ and $\varepsilon > 0$ there is $x' \in \Gamma(H)$ such that $\|x(t) - x'(t)\| < \varepsilon$ for all t in some neighbourhood of t_0, then $x \in \Gamma(H)$.

Then $\Gamma(H)$ is a Hilbert $C(M)$-module where (for $x \in \Gamma(H),\ a \in C(M),\ t \in M$):

$$(x \cdot a)(t) := a(t)x(t), \qquad \langle x, y\rangle_{C(M)}(t) := \langle x(t), y(t)\rangle.$$

Every Hilbert $C(M)$-module is of the form described above!!!

3) Maps on Hilbert C^*-modules

Def.

Let X and Y be Hilbert A-modules. We say that a map $T : X \to Y$ is an **adjointable operator** if there exists a map $T^* : Y \to X$ such that

$$\langle Tx, y\rangle_A = \langle x, T^*y\rangle_A, \qquad \text{for all } x \in X,\ y \in Y.$$

It follows (exercise) that both T and T^* are bound $\mathbb{C}$-linear and A-linear operators. Moreover, T determines uniquely T^* and vice versa.

$\mathcal{L}(X, Y) := \{T : X \to Y \text{ adjointable}\}$ is a Banach subspace of $B(X, Y)$,

$\mathcal{L}(X) := \mathcal{L}(X, X)$ is a unital C^*-algebra (exercise).

Ex. (Not every bounded A-linear map is adjointable)

Let $A = C(M)$ where M compact Hausdorff and $J := \{a \in C(M) : a(t_0) = 0\}$ where $t_0 \in M$ is a non-isolated point. The inclusion map $T : J_A \to A_A$ is isometric, A-linear but NOT adjointable. If T were adjointable, then

$$x^* = \langle Tx, 1\rangle_A = \langle x, T^*(1)\rangle_A = x^* T^*(1), \qquad \text{for all } x \in J,$$

which implies that $T^*(1)(t) = 1$ for $t \in M \setminus \{t_0\}$ and $T^*(1)(t_0) = 0$. $\lightning$

Let $x \in X$, $y \in Y$. The map $\Theta_{x,y} : Y \to X$ defined by

$$\Theta_{x,y}(z) = x\langle y, z\rangle_A$$

is an adjointable operator with $\Theta_{x,y}^* = \Theta_{y,x}$.

Def

Elements of $\mathcal{K}(Y,X) := \overline{\text{span}}\{\Theta_{x,y} : x \in X, Y \in Y\}$ are called (generalized) **compact operators** from Y to X.

The space $\mathcal{K}(X) := \mathcal{K}(X,X)$ is a (closed two-sided) ideal in $\mathcal{L}(X)$.

Ex.1 (Hilbert spaces) If $A = \mathbb{C}$, then $X = H$, $\mathcal{L}(X) = B(H)$ and $\mathcal{K}(X) = K(H)$.

Ex.2 (C^*-algebras) If A a C^*-algebra, then $\mathcal{K}(A_A) \cong A$ and $\mathcal{L}(A_A) \cong M(A)$ is the multiplier algebra of A - maximal essential unitization of A.

Rem. For every Hilbert A-module X we have $M(\mathcal{K}(X)) \cong \mathcal{L}(X)$.

4) C^*-correspondences

Let A, B be C^*-algebras.

Def.

C^***-correspondence from** A **to** B is a (right) Hilbert B-module X equipped with homomorphism $\phi_X : A \to \mathcal{L}(X)$ - left action of A on X. We write $a \cdot x := \phi_X(a)x$.

$$A \xrightarrow{X} B$$

We say that

- X is *faithful* if ϕ_X is faithful
- X is *nondegenerate* if $\phi_X(A)X = X$
- X is *proper* if $\phi_X(A) \subseteq \mathcal{K}(X)$

Ex.1 $\{\text{Representations } \pi : A \to B(H)\} = \left\{ \begin{array}{c} C^*\text{-correspondences from } A \text{ to } \mathbb{C} \\ A \xrightarrow{H_\pi} \mathbb{C} \end{array} \right\}$

Ex.2 (Homomorphisms) If $\alpha : A \to B$ a $*$-homomorphism then $A \xrightarrow{X_\alpha} B$ where $X_\alpha := \alpha(A)B$ is equipped with operations (where $x, y \in X_\alpha,\ a \in A,\ b \in B$):

$$a \cdot x := \alpha(a)x, \qquad x \cdot b := xb, \qquad \langle x, y \rangle_B := x^* y$$

Ex.3 (Concrete C^*-correspondences) Let $X \subseteq B(H)$ be a closed linear space and $A, B \subseteq B(H)$ are C^*-subalgebras such that

$$XB \subseteq X, \qquad X^*X \subseteq B, \qquad AX \subseteq X.$$

Then X is naturally a C^*-correspondence from A to B.
Every C^*-correspondence can be represented in this form!!!

Ex.4 (C^*-correspondences vs graphs) Let V, W be sets. Let $G = (E, s, r)$ be a graph from V to W, i.e. E is a set and $s : E \to V$ and $r : E \to W$ are maps. We define C^*-correspondence X_G from $A = C_0(W)$ to $B := C_0(V)$ by:

$$X_G := \{x \in C_0(E) : V \ni v \longmapsto \sum_{e \in s^{-1}(v)} |x(e)|^2 \in \mathbb{C} \text{ is in } C_0(V)\},$$

$$\langle x, y \rangle_A(v) := \sum_{e \in s^{-1}(v)} \overline{x(e)} y(e),$$

$$(a \cdot x)(e) := a(r(e))x(e),$$

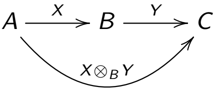
$$(x \cdot b)(e) := x(e)b(s(e)).$$

Every C^*-correspondence from $C_0(W)$ to $C_0(V)$ is of this form!!!

Def. (Tensor product)

If $A \xrightarrow{X} B$ and $B \xrightarrow{Y} C$ then there is (exercise) a C^*-correspondence $A \xrightarrow{X \otimes_B Y} C$ where $X \otimes_B Y = \overline{\text{span}}\{x \otimes y : x, \in X, y \in Y\}$ and

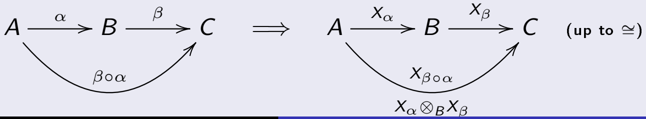
$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_C = \langle y_1, \langle x_1, x_2 \rangle_B \cdot y_2 \rangle_C .$$



Ex.1 (Induced representations) If $A \xrightarrow{X} B$ is a C^*-correspondence and $B \xrightarrow{H_\pi} \mathbb{C}$ is a representation of B, then $A \xrightarrow{X \otimes_B H_\pi} \mathbb{C}$ is a representation of A.

Usually it is denoted by $X - \text{Ind}_B^A \pi$ and the underlying Hilbert space by $X \otimes_\pi H$

Ex.2 (Composition of homomorphisms) If $\alpha : A \to B$ and $\beta : B \to C$ are $*$-homomorphisms then $X_\alpha \otimes_B X_\beta \cong X_{\beta \circ \alpha}$ where $\beta \circ \alpha : A \to C$.



Ex.3 (Concrete tensor products) Let $A, B, C \subseteq B(H)$ and $X, Y \subseteq B(H)$ be concrete C^*-correspondences $A \xrightarrow{X} B$ and $B \xrightarrow{Y} C$:

$$XB \subseteq X, \quad X^*X \subseteq B, \quad AX \subseteq X \quad \text{and} \quad YC \subseteq Y, \quad Y^*Y \subseteq C, \quad BY \subseteq Y.$$

Then $\overline{XY} = \overline{\text{span}}\{xy : x \in X, y \in Y\} \subseteq B(H)$ is a concrete $A \xrightarrow{\overline{XY}} C$:

$$\overline{XY}C \subseteq \overline{XY}, \quad (\overline{XY})^*\overline{XY} \subseteq C, \quad A\overline{XY} \subseteq \overline{XY}$$

and

$$X \otimes_B Y \cong \overline{XY}.$$

Ex.4 (C^*-correspondences vs graphs) Let $G = (E, s, r)$ a graph from V to W and $H = (F, s, r)$ a graph from W to U. Define the graph

$$H \circ G := (F \circ E, s, r)$$

where $F \circ E := \{(f, e) \in F \times E : s(f) = r(e)\}$, $s(f, e) := s(e)$, $r(f, e) = r(f)$. Then

$$X_H \otimes_B X_G \cong X_{H \circ G}$$

Category C^*-$\mathfrak{alg}$

Objects $\equiv$ C^*-algebras   Morphisms $\equiv$ $*$-homomorphisms

"Category" C^*-$\mathfrak{corr}$

Objects $\equiv$ C^*-algebras

Morphisms $\equiv$ nondegenerate C^*-correspondences

Associativity: If $A \xrightarrow{X} B$, $B \xrightarrow{Y} C$ and $C \xrightarrow{Z} D$, then

$$X \otimes_B (Y \otimes_C Z) \cong (X \otimes_B Y) \otimes_C Z$$

Identity elements = C^*-algebras: If $A \xrightarrow{X} B$, then

$$X \otimes_B B \cong X \qquad (A \otimes_A X) \cong X$$

Invertible elements: $A \xrightarrow{X} B$ is invertible if there is $B \xrightarrow{X^\star} A$

$$X^\star \otimes_A X \cong B \qquad X \otimes_B X^\star \cong A$$

This holds if and only if X is (Morita-Rieffel) equivalence bimodule.

5) Hilbert C^*-bimodules

Def.

X is a **Hilbert A-B-bimodule** if X is a right Hilbert B-module and a left Hilbert A-module such that the respective inner products satisfy

$$_A\langle x, y\rangle z = x\langle y, z\rangle_B, \qquad x, y, z \in X.$$

Rem.1. Every right Hilbert B-module is a Hilbert $\mathcal{K}(X)$-B-bimodule where

$$_{\mathcal{K}(X)}\langle x, y\rangle := \Theta_{x,y}, \qquad x, y \in X.$$

Rem.2. Every Hilbert A-B-bimodule X is a C^*-correspondence from A to B.

The adjoint B-A-bimodule $X^\star$ is a C^*-correspondence from B to A and

$$X \otimes_B X^\star \cong \overline{\langle X, X\rangle}_B \qquad (X^\star \otimes_A X) \cong {}_A\overline{\langle X, X\rangle}$$

where

$$\overline{\langle X, X\rangle}_B := \overline{\text{span}}\{\langle x, y\rangle_B : x, y \in X\} \text{ is an ideal in } B$$

$${}_A\overline{\langle X, X\rangle} := \overline{\text{span}}\{{}_A\langle x, y\rangle : x, y \in X\} \text{ is an ideal in } A.$$

Def.

X is a (Morita-Rieffel) **equivalence** A-B-**bimodule** if

1) X is a Hilbert A-B-bimodule
2) $\overline{\langle X, X\rangle_B} = B$ and $\overline{{}_A\langle X, X\rangle} = A$.

If such X exists we say that A and B are **Morita equivalent**.

Rem. Every Hilbert A-B-bimodule is an equivalence $\overline{{}_A\langle X, X\rangle}$-$\overline{\langle X, X\rangle_B}$-bimodule.

Ex.1 Hilbert space H establishes Morita equivalence between $\mathbb{C}$ and $K(H)$.

Ex.2 Let $p \in C$ where C^*-algebra. The right ideal $X := pC$ is an equivalence bimodule from the hereditary algebra $A := pCp$ to the ideal $B := CpC$.

In general

"X is a Hilbert A-B-bimodule $\iff \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is a C^*-algebra"

C^*-algebras A and B are Morita equivalent $\iff$ they can be embedded into a C^*-algebra C as full and complementary corners.

Thm. (Brown, Green, Rieffel 1997)

If A and B have countable approximate units then

$$A, B \text{ are Morita equivalent} \quad \Longleftrightarrow \quad A \otimes K(H) \cong B \otimes K(H).$$

Thm.

If A and B are Morita equivalent then

1) $\widehat{A} \cong \widehat{B}$
2) $\mathrm{Ideal}(A) \cong \mathrm{Ideal}(B)$
3) A is nuclear if and only if B is nuclear
4) A and B have the same K-theory
5) ...

6) Cuntz-Pimsner C^*-algebras

Def: Let $A \xrightarrow{X} A$ be a C^*-correspondence from A to A.

Representation of X in a C^*-algebra C is a pair (π, ψ) where $\pi : A \to C$ is a $*$-homomorphism, and $\psi : X \to C$ is such that

$$\pi(a)\psi(x) = \psi(ax), \qquad \psi(x)\pi(a) = \psi(xa), \qquad \psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A)$$

When π is injective, we say that (π, ψ) is **injective**.

Consider tensor powers $X^{\otimes n}$ of X ($X^{\otimes 0} := A$) and Hilbert A-module direct sum

$$\mathcal{F}(X) := \bigoplus_{n=0}^{\infty} X^{\otimes n} = \{(x_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} \langle x_n, x_n\rangle_A < \infty\}$$

It carries a diagonal left action $\pi : A \to \mathcal{L}(\mathcal{F}(X))$ where

$$\pi(a)(x) := ax \qquad x \in X^{\otimes n}.$$

We have a map $T : X \to \mathcal{L}(\mathcal{F}(X))$ where for $x \in X$, $T(x)$ is a 'creation operator':

$$T(x)y = \begin{cases} xy & \text{if } y \in X^{\otimes 0} = A \\ x \otimes_A y & \text{if } y \in X^{\otimes n} \text{ for } n \geq 1 \end{cases}, \qquad T(x) : X^{\otimes n} \to X^{\otimes n+1}$$

(π, T) is an injective representation of the C^*-correspondence X.

Def. (Pimsner 1997, Katsura 2003)

The C^*-algebra $\mathcal{T}_X$ generated by $\pi(A)$ and $T(X)$ is called **Toeplitz algebra of** X.

The **Cuntz-Pimsner algebra of** X is

$$\mathcal{O}_X := \mathcal{T}_X / \mathcal{J}_X$$

where $\mathcal{J}_X$ is the largest ideal in $\mathcal{T}_X$ such that (π, T) factors through to an injective representation of X in $\mathcal{T}_X/\mathcal{J}_X$ and $\mathcal{J}_X$ is gauge invariant.

Rem. Recall that $\phi_X : A \to \mathcal{L}(X)$ is the left action homomorphism. The ideal

$$J_X = (\ker \phi_X)^{\perp} \cap \phi_X^{-1}(\mathcal{K}(X))$$

induces the ideal $\mathcal{J}_X := \mathcal{K}(\mathcal{F}(X)J_X)$ in $\mathcal{L}(\mathcal{F}(X))$ and we have $\mathcal{J}_X \subseteq \mathcal{T}_X$.

Ex.1 (Cuntz algebras) If $X = H$ is a Hilbert space and $d = \dim(H)$, then

$$\mathcal{T}_X = C^*(S_1, ..., S_d : S_i^* S_j = 1\delta_{i,j},\ i, j = 1, ..., d)$$

$$\mathcal{O}_X = \mathcal{O}_d \text{ Cuntz algebra if } d \geq 2 \ \text{ and } \ \mathcal{O}_{\mathbb{C}} = \mathbb{T}$$

Ex.2 (graph algebras) If $A = C_0(V)$ where V discrete then $X = X_G$ where $G = (E, s, r)$ is a directed graph from V to V, and

$$\mathcal{O}_X = \mathcal{O}_G \text{ – the graph } C^*\text{-algebra}$$

It is a universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in V\}$ and partial isometries $\{s_e : e \in E\}$ subject to relations

$$s_e^* s_e = p_{s(e)}, \quad s_e s_e^* \leq p_{r(e)} \quad \text{and} \quad p_v = \sum_{v \in r^{-1}(e)} s_e s_e^* \text{ if } 0 < |r^{-1}(e)| < \infty$$

Ex.3 (crossed products by endomorphisms)
If $\alpha : A \to A$ an endomorphism then

$$\mathcal{O}_{X_\alpha} = A \rtimes_\alpha \mathbb{N} \text{ – the crossed product}$$

If $\alpha : A \to A$ injective, and A unital then

$$A \rtimes_\alpha \mathbb{N} = C^*(A \cup \{U\} : U^* U = 1, \alpha(a) = UaU^* \text{ for all } a \in A)$$